This course is about Hecke algebras, their representations, and their role in helping to understand an important class of representations of matrix gps.

A representation of $G$ (over $\mathbb{R}$: comm. ring with unit) is a group hom: $\rho : G \to \text{GL}(V)$, $V : \mathbb{R}$-module.

(familiar example: $\mathbb{R} = \mathbb{C}$, $V = \mathbb{C}^n$, then $\text{GL}(\mathbb{C}^n)$: invertible $n \times n$ matrices)


$\rightarrow \quad \left( \sum \alpha g \cdot \cdot g \right) \cdot v \overset{\text{def}}{=} \sum \alpha g \rho(g) \cdot v$)

Given \( \rho \).

Hecke algebra is deformation of gp. algebra.

Two definitions — one as formal presentation using Coxeter gps and one as fractions on certain gp.

Coxeter gp: Specify gp amel set of generators $S = \{ s_1, \ldots, s_r \}$ order $2$ elements $W$.

now just explain order of $s_i s_j$ for all pairs $i \neq j$.

Often encode this information in r X r matrix or in a graph.
(Induction) finite Coxeter groups are classified:

4 infinite families

\[ A_{n-1} : \begin{array}{c}
\vdots \\
3 & 3 & \cdots & 3 \\
1 & 2 & \cdots & n-1 \\
\end{array} \quad I_2(m) : \begin{array}{c}
1 \\
2 \\
m \\
\end{array} \]

\[ B_n : \begin{array}{c}
0 & 1 & 2 & \cdots & n-1 \\
4 & 3 \\
\end{array} \]

\[ D_n : \begin{array}{c}
1 & 2 & 3 & \cdots & n-1 \\
\end{array} \]

except that types:

\[ E_6, F_4, H_4 \]

(6 of these)

but we may use infinite Coxeter groups as well (like \( A_2^{(1)} \))

Back to example. Mention that it has a basis \( T_w \) \( w \in W \).

\[ T_{s_0} \cdot T_w = T_{sw} \text{ if } l(sw) > l(w) \]

\[ T_{s_0} \cdot T_w = T_{sw} + (g-1)T_w \text{ if } l(sw) < l(w). \]

\( l : W \rightarrow \mathbb{Z}_{\geq 0} \) : length function. Write \( w = s_i \cdots s_p \) then with \( p \) minimal.

then \( l(w) \overset{\text{def}}{=} p \).
Hecke algebras defined in two ways -

1. Give a presentation for them in generators, relations associated to a Coxeter group (reflection group) $W$ generated by order 2 elements and relations.

2. Express as a space of functions on double cosets in an algebraic group (familiar matrix group).

Example: $W = S_3 = \langle b_1 = (12), b_2 = (23) \mid b_1^2 = b_2^2 = 1, (b_1b_2)^3 = 1 \rangle$

Then we define Hecke algebra over the commutative field $F$ containing parameter $q$. E.g., $F = \mathbb{Q}[q, q^{-1}]$ view $q$ as formal parameter.

With generators $T_1, T_2$

with $(T_1T_2)^2 = 1$ and

$T_i^2 = (q-1)T_i + q$.

Call this $H_q[S_3]$.

Note if $q = 1$, then quadratic relation reduces to $T_i^2 = 1$, just the group algebra $\mathbb{Z}[S_3]$ or $\mathbb{C}[S_3]$.

(Obtained using mult. in gp, extending linearly.)

Goal 1: Understand representations of $H_q[W]$ for wide class of Coxeter gp $W$. Remember representations of gp. algebra $\leftrightarrow$ representations of underlying gp.
Second incarnation: \( G = \text{GL}_2(\mathbb{F}_q) \), \( q \): prime power \( p^k \).

\[ B = \text{standard Borel subgroup} \]
\[ = \text{upper triangular matrices in } G \]
\[ \text{with coefficients in } \mathbb{F}_q. \]

Consider "double cosets" \( BgB = \{ b_1gb_2 \mid b_1, b_2 \in B \} \).

They partition \( G \), and in our case, it is known that

\[ G = \bigsqcup BwB \quad \text{("Bruhat decomposition")} \]

We say viewed in \( G \) as permutation matrices.

Consider functions on \( B \backslash G / B \).

\[ T_w : \text{characteristic function of } BwB. \]

How to make algebra:

\[ T_w * T_u = \sum_{v \in S_3} m_{u,v} T_v \]

where \( m_{u,v} = \# \text{ of cosets } Bx \)

Recall that given gp. \( G \),

define convolution of two functions \( \phi, \psi \) by

\[ (\phi * \psi)(g) = \frac{1}{|G|} \sum_{x \in G} \phi(x) \psi(x^{-1} g) \]

\[ = \frac{1}{|G|} \sum_{x \in G} \phi(gx) \psi(x^{-1}) \]

\( \phi * \psi \) to realize this multiplication.

\[ \text{normalize convolution } \phi \text{ replacing } \frac{1}{|G|} \text{ by } \frac{1}{|G|} \]
Theorem (Iwahori) Two incarnations are isomorphic algebras.

Why important to us:

Theorem (Titik) Representations of $H^*_G \rightarrow \text{Rep}_B$ of $G(\mathbb{F}_q)$ with $B$-fixed vector

$V^B \leftrightarrow V$

General theory of representations of $G(\mathbb{F}_q)$ rather difficult.

Annals of Math '76 – Deligne-Lusztig (prove conj. of Macdonald)

Interesting groups in this collection – families associated to Coxeter groups $A_1, B_1, \ldots, G$. Eventually: (Iwahori-Matsumoto)

Given affine Coxeter gp., associate Hecke algebra, relate this to space of $I$-bi-invariant $\phi$ (smooth, comp. supp. functions on group $G(\mathbb{Q}_p)$, $\mathbb{Q}_p$: $p$-adic field, $I$ (What is $I$?)

Perform whole story over again...