For gp. algebra, we found that map: \( \tau(h) = c_0(h) \) coeff. of identity in basis of gp. alg.
gave us symmetrizing trace that recovered orthogonality relations from gp. repn theory.

Other traces: \( V: H\)-module, free over \( R\)-ring for \( H\), then 
\[ X_V : H \to R \]
\[ h \mapsto \text{Tr}(p_V(h)) \]

Very early theorem: \( H\) semisimple \( \iff \) trace from \( R\) for regular repn gives rise to non-degenerate pairing.

Existence of trace functions leads to orthogonality relations.

But nicer choices can make for easier computations.

Do more - give formulas for idempotents using "Schur elements".

Integral closure assumption - guarantees that character table takes values in \( R\) on \( R\) obvious generalization

of gp. alg. trace.

Nice trace on Hecke algebra: \( \tau: H \to R \) defined by 
\[ \tau(T_e) = 1 \]
\[ \tau(T_w) = 0 \] if \( w \neq e \) in \( W \)

Claim: \[ \tau(T_w T_{w'}) = \sum g_w \text{ if } w^{-1} = w' \]
\[ 0 \text{ else} \]

With \( g_w := g_{s_1} \cdots g_{s_k} \) when \( w = s_1 \cdots s_k \) e.g. \( g_e = 1 \) if all \( g_{s_i} = g_s \). \( g_e = 1 \).

If \( g_s \)'s units in \( R\), then \( \tau\) is symmetrizing trace with basis dual to \( \frac{1}{2} T_{w}^2 \) given by 
\[ T_w := g_w^{-1} \cdot T_{w^{-1}}. \]
pf: the formula for the product is proved by induction on length. Base case \( l(w) = 0 \) is \( w = e \), then clear. Usual cases depending on \( l(cs_{w'}) > l(w') \) or not.

Check in rank one: \( W = \langle s \rangle \).

\[
\tau(T_s^2) = \tau(q_s^{-1}T_s + q_sT_e)
\]

\[
= q_s \tau(T_e) = q_s. \checkmark
\]

to see it is a trace function, note \( q_w = q_{w^{-1}} \) since if \( W = s_1 \ldots s_k \) then \( w^{-1} \) given by reduced writing \( s_i \)'s in reverse order

So \( \tau(T_{wT_{w'}}) = q_w \) if \( w^{-1} = w' \) \( \checkmark \) as \( w^{-1} = w' \)

\( \tau(T_{wT'_w}) = q_{w'} \) if \( (w')^{-1} = w \) iff \( (w')^{-1} = w' \)

Now taking \( T_w = q_{w^{-1}}T_{w^{-1}} \), then this is basis

and \( \tau(T_{wT'_w}) = q_{w'}^{-1} \tau(T_{w^{-1}T_{w'}}) = \left\{ \begin{array}{ll} 1 & \text{if } w' = w \\ 0 & \text{else} \end{array} \right. \)

So indeed \( \tau \) is non-deg. with dual basis as above.

Orthogonality relations in Hecke's clear:

\[
\sum_{b \in \mathfrak{g}_S} \chi_v(b) \chi_{v'}(b') = \left\{ \begin{array}{ll} \sum_{v \in \dim_k V} & \text{if } \chi_v = \chi_{v'} \\ 0 & \text{else} \end{array} \right. \]

\( H / K \): splitting field

\[
\sum_{w \in W} q_{w^{-1}}^{-l(w)} \chi(T_w) \chi'(T_{w^{-1}}) = \left\{ \begin{array}{ll} \sum_{v \in \dim_k V} & \text{if } \chi_v = \chi_v' \\ 0 & \text{else} \end{array} \right. \]

\( c_{\mu_{tw''}} = \sum_{w \in W} q_{w^{-1}}^{-l(w)} q_{l(w)} q_{l(w)} q_{l(w)} = \sum_{w \in W} q_{l(w)} \): Poincaré poly. of \( W \).
Strongly conjugate: \( W \): finite Coxeter group.

First two elements are "elementarily strongly conjugate" if \( l(W) = l(W') \)

and \( \exists x \in W \) s.t. \( wx = xw' \) and \( l(wx) = l(w) + l(x) \)

or \( xw = w'x \) and \( l(xw) = l(x) + l(w) \)

"strongly conjugate" if \( \exists \) sequence \( W = W_0, W_1, \ldots, W_r = W' \) with \( W_i, W_{i+1} \) elementarily strongly conjugate.

Thus (3.2.9 in Geck-Pfeiffer?) Any two elements in \( C_{\text{min}} \) are strongly conjugate.

(\( C_{\text{min}} \): elts of min. length in a given conj. class \( C \) of \( W \))

Weaker notion from conjugacy graph - graph with vertices \( \{W : w \neq \text{new} \} \)

and labeled directed edges \( W \rightarrow W' \) if \( W' = wsW \) with \( l(W') \leq l(W) \).

E.g. \( A_2 \):

- Define \( \text{Cyc}(w) = \{ v \in W | \exists \text{ path in conj. graph} \text{ from } w \to v, v \to W' \} \)

- "cyclic shift class" (imply \( l(w) = l(v) \))

If \( W, W' \) satisfy \( W' = sW \). Need either \( l(sw) > l(w) \) or \( l(ws) > l(w) \)

to be strongly conjugate.

Problem case: \( l(sw) = l(ws) < l(w) \). Claim: then \( W = sW \) (= \( W' \)) \( \uparrow \).

Thus:
(a) Given \( w \in C \), \( \exists W' \in C_{\text{min}} \) with \( W \rightarrow W' \)

(b) Given \( w, w' \in C_{\text{min}} \), \( \exists W' \in \text{Cyc}(w) \) and \( x \in W \) s.t. \( W', W \) elementarily strongly conj. via \( x \).
Lemma: If $w, w' \in W$ are strongly conjugate, then $Tw$ and $Tw'$ are conjugate in $H$ and hence $Tw \equiv Tw'$ mod $[H, H]$. 

**pf:** if $h' = xh x^{-1}$ for some unit $x \in H$ then 

$$h - h' = h - xh x^{-1} = x^{-1} x h - x h^{-1} = [x^{-1}, xh] \in [H, H]$$

so it is enough to show they are conjugate.

**claim:** if $w, w' \in W$ are strongly conjugate, then they are conjugate in the braid group. (Recall braid gp is generated by $s_i \in S$ with

$$\alpha \beta \beta \alpha \ldots = \beta \ldots \alpha$$

most factors

So if $\langle s \rangle$, infinite cyclic gp - integer powers of $s$

then, if we can show claim, we're in good shape as $f$ algebra homomorphism

from $\mathbb{R}[\text{Braid gp}] \rightarrow H$ so conjugation is preserved.

$$w \mapsto Tw$$

(so we are using fact that $Tw$ invertible here, which requires $g$ invertible.

In general Hecke algebra, adjoin $g_i$ and $\bar{g}_i$.)

**pf of claim:** Suffices to show if $w_1, w_2$ are elementarily strongly conjugate,

But if $x \in W$ s.t. $w_1 x = x w_2$ and $l(w_1 x) = l(w_1) + l(x)$

this is precisely what is needed for equality $w_1 x = x w_2$ in $B$; braid gp.

(can't use quadratic relation $s^2 = 1$ anywhere)