Strongly conjugate: \( W \): finite Coxeter group.

First two elements are "elementarily strongly conjugate" if

\[
W, W' \in W
\]

and \( x \in W \) s.t.

\[
\begin{align*}
Wx &= xW' & \text{and } l(wx) &= l(w) + l(x) \\
\text{or } xW &= W'x & \text{and } l(xw) &= l(x) + l(w)
\end{align*}
\]

"strongly conjugate" if \( \exists \) sequence \( W = w_0, w_1, \ldots, w_r = W' \) with \( \{ w_i \mid w_i \text{ elementarily strongly conjugate} \} \).

Thm. (3.2.9 in Geck-Pfeiffer) Any two elements in \( C_{\text{min}} \) are strongly conjugate.

(\( C_{\text{min}} \): elts of min length in a given conj. class \( C \) of \( W \)).

Weaker notion from conjugacy graph - graph with vertices \( \forall w \exists w' \in W \)

and labeled directed edges \( w \xrightarrow{\sigma} w' \) if \( w' = \sigma w \sigma^{-1} \) with \( l(w') \leq l(w) \).

\[ s_2s_1 = s_1s_2 = s_2s_1s_2 \]

Define \( \text{Cyc}(w) = \{ v \in W \mid w \xrightarrow{\sigma} v, \sigma \xrightarrow{\sigma} w \} \) (implies \( l(w) = l(v) \))

"cyclic shift class" a path exists in conj. graph.

If \( \sigma w \sigma \) satisfy \( w' = \sigma w \sigma \). Need either \( l(sw) > l(w) \) or \( l(ws) > l(w) \) to be strongly conjugate.

Problem case:

\[
l(sw) = l(ws) < l(w). \quad \text{claim: then } w = sws (= w') \Rightarrow.
\]

Thm: (a) Given \( w \in C \), \( \exists w' \in C_{\text{min}} \) with \( w \xrightarrow{} w' \)

(b) Given \( w, w' \in C_{\text{min}} \), \( \exists w' \in \text{Cyc}(w) \) and \( x \in W \) s.t. \( w', v \) elementarily strongly conj. via \( x \).
**Lemma:** If \( w, w' \in W \) are strongly conjugate, then \( Tw \) and \( Tw' \) are conjugate in \( H \) and hence \( Tw \equiv Tw' \mod [H, H] \).

**pf:** if \( h' = xhx^{-1} \) for some unit \( x \in H \) then
\[
h - h' = h - xhx^{-1} = x^{-1}xh - xhx^{-1} = [x^{-1}, xh] \in [H, H]
\]
so it is enough to show they are conjugate.

**Claim:** If \( w, w' \in W \) are strongly conjugate, then they are conjugate in the braid group. (Recall braid gp is generated by \( s \in S \) with
\[
sts\ldots = tsts\ldots
\]
max. \( m \)st factors
factors
So if \( \langle s \rangle \), infinite cyclic gp - integer powers of \( s \)

then, if we can show claim, we're in good shape as \( J \) algebra homomorphism
\[
R[Braid gp] \to H \quad \text{so conjugation is preserved}
\]
\( W \leftrightarrow Tw \)

(so we are using fact that \( Tw \) invertible here, which requires \( g \), invertible.
In generic Hecke algebra, adjoin \( g \) and \( g^{-1} \).
)

**pf of claim:** Suffices to show if \( w_1, w_2 \) are elementarily strongly conjugate.

But if \( x \in W \) s.t. \( w_1x = xw_2 \) and \( \ell(w_1x) = \ell(w_1) + \ell(x) \)
this is precisely what is needed for equality \( w_1x = xw_2 \) in \( B \) braid.

(aren't we using quadratic relation \( s^2 = 1 \) anywhere?)
Now given \( w \in W \), either \( w \in C_{\text{min}} \subseteq C \), and then

\[
T_w \equiv T_{w_0} \pmod{[H,H]},
\]

or else \( w \in C \setminus C_{\text{min}} \), and then \( w \) related to \( w' \) by \( \gamma \)-map \( \psi \) in \( C_{\text{min}} \).

(sequence \( w = w_0, \ldots, w_r = w' \) with \( \ell(w_i) \leq \ell(w_{i-1}) \))

with \( w_{i-1} = w_i \)

\( W = w_0 \) and \( w_i = s_{w_i} s \) with \( \ell(w_i) \leq \ell(w) \).

Suppose \( W = w_0, \ldots, w_i, w_{i+1} \) equality first drop.

\[ \text{must be strict somewhere since } \ell(w') < \ell(w). \]

claim: \( T_w = q_s T_{w_i} s \pmod{[H,H]} \)

\[ + (q_s - 1) T_{w_i} \]

Indeed had \( T_w = T_{w_i} \)

now \( T_{w_i} \)

\[ (q_s T_{w_i} s + (q_s - 1) T_{w_i}) T_s \]

\[ = q_s T_{w_i} s + (q - 1) T_{w_i} = T_s \cdot T_{w_i} \]

so inverting \( T_s \), we're done.

Example: \( W = s_1 s_2 s_1 \in A_2 \). \( W^* = s_2, s = s_1 \)

then \( T_w = q_s T_{s_2} + (q - 1) T_{s_1 s_2} \)

character table example...
$A_2$ character table

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>s</th>
<th>st</th>
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<tbody>
<tr>
<td>triv</td>
<td>1, $q$</td>
<td>$q^2$</td>
<td></td>
</tr>
<tr>
<td>sgn</td>
<td>1, -1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>reflection</td>
<td>2, $q-1$, $-q$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Know, for example that all other 3 elements are in $C_{min} = \equiv 0 \mod \langle H_i, H_j \rangle$ except for st's = wo.

Our algorithm gave $T_W = q T_s + (q-1) T_{ts}$ $[H_i, H_j]$

So $T_W$ in reflection rep'n $p$ has $Tr(p(T_W))$

Orthogonality: used our nice trace function

to conclude:

$$
\sum_{w \in W} q^{-\ell(w)} \chi(T_w) \chi'(T_{w^{-1}}) = \sum_{\chi \neq \chi'} \chi(T_e) \cdot \delta_{\chi, \chi'}
$$

= 0.

Check: for reflection rep'n:

$$
4 + 2 \left( \frac{q-1}{q} \right)^2 + 2 \left( \frac{-q}{q} \right)^2 / 2
$$

= $4 + 2 (q-2 + q^{-1}) + 2 / 2$

= $2 (q^2 + 1 + q^{-1}) / 2$

= $q^2 + 1 + q^{-1}$. 


Induction from parabolic subalgs. Punchline - induction compatible with specialization.

Just as with Coxeter gps, given

subset \( J \subseteq S \), with \( W_J \leq W \) the parabolic subgp.,

form subalgebra \( H_J = \langle T_w \mid w \in W_J \rangle_R \) : free mod. \(/R\)

(If we define \( X_J := \{ w \in W \mid l(sw) > l(w) \forall s \in J \} \)

then \( W = W_J \times X_J \) and if \( W = (v, x) \)

then \( l(w) = l(v) + l(x) \).

so have nice set of right coset reps

\( \star \) of \( W_J \) in \( W \).

(\#)

Deodhar's Lemma: \( J \subseteq S \), \( x \in X_J \), \( s \in S \), then either

\( xs \in X_J \) or \( xs = ux \) for some \( u \in J \). (So \( u \)'s simple.)

(if \( xs \notin X_J \), \( J \subseteq J \) s.t. \( l(uxs) < l(xs) \)

since \( x \in X_J \), then \( l(uxs) > l(x) \)

\( \Rightarrow \) \( l(xs) > l(x) \), use cancellation law to

show \( xs = ux \).

(\#)

so \( H = \bigoplus_{x \in X_J} H_J \cdot T_x \).
Define \[ \text{Ind}_{H_j}^H (v) := v \otimes_{H_j} H. \]

\[ (v \otimes h) \cdot h' = v \otimes (hh') \]

So in Hecke algebra, elements in \[ \text{Ind}_{H_j}^H (v) \]

may be written uniquely in the form \[ \sum_{x \in x_j} v_x \otimes T_x \]

some \( v_x \)'s \( \in V \).

How do we act on \( \text{elts by } T_s \)?

\[ (v \otimes T_x) \cdot T_s = \]

\[ v \cdot T_x \otimes T_s \]

if \( xs = tx \)

\[ v \otimes T_{xs} \]

if \( xs \in x_j \)

\[ v \otimes (qT_{xs} + (q-1)T_x) \]

if \( xs \notin x_j \)

\[ \sum_{t \in J} v \cdot T_t \otimes T_x \]

\[ v \otimes T_{xs} \]

\[ \text{if } xs \in x_j \]

\[ \text{if } xs \notin x_j \]

Use Dedekind's Lemma

if \( V \) is free \( R \)-mod.

Then we can also expand \( v \cdot T_s \) action

in terms of basis.

Example: \( A_3 = \langle s_1, s_2, s_3 \rangle \)

\[ \text{Ind}_{H_j}^H \text{ (triv)} \]

with \( J = \langle s_1, s_2 \rangle \)

\[ |x_J| = 4 - \]

\[ x_J = \{ 1, s_1, s_3, s_2s_1, s_3s_2s_1 \} \]

\( \text{triv acts on 1-dim'\'} \)

\[ \text{v.s. } C \]

with gen 1.