In the middle of pf of Snowden's rule, computing $\chi(T_{W_0})$, $\chi \in \text{Irr}(KH)$ and $W_{[n]}$ : min in class $C_{[n]}$.

Two part plan. 1. Showed $T_{W_0}^2$ is central in $H$, so acts by scalar on $V$ with char $\chi$ by $\chi = \prod_{s \in S'} g_s, \text{ s.e.T}, = N_s \cdot (1 + \frac{\chi_s(s)}{\chi_s(1)}$ equivalence classes under conj.

2. If $W_c$ is Coxeter elt in $W$, order $h$ then $W_h = W_0^2$ in braid monoid, hence $T_{W_c} = T_{W_0}$ in $H$, so can determine $\chi(T_{W_c})$.

Notes:
- Follows Geck's pf. [99 Complts' Radix]
- Mentions feat it is open to give analogous formula for other types.
- Sneaky thing we did — reinterpret $\chi(T_{W_c}) = q^*$ as $\chi_1(w) \cdot \det(...)$ with good functional properties.
- Splitting field for generic Hecke algebra (Benson-Curtis, Lusztig)
- Any field containing $K(\sqrt{q_s} | s \in S)$ where $K \supset \mathbb{Q}(\cos(\frac{2\pi}{n_s} m_{sd} | s \in S))$
Proposition: In braid monoid \( W_c^h = w_0^2 \) where \( W_c : \) Coxeter element and \( h \) is its order.

**Proof:** Have short list of (families of) finite Coxetergps.
Examine Dynkin diagrams, notice that we can partition (uniquely) \( S = S_1 \cup S_2 \) s.t. all els of \( S_1 \) commute.

**Example:** \( A_{n-1} \), take \( S_1 : \) ?odd #d vertices ?, \( S_2 : \) ?even #d vertices ?
write \( W_c = w_1 w_2 \) (Better choose Coxeter element \( W_0 \) so that it respects this decomposition \( S = S_1 \cup S_2 \)).

**Claim:** \( w_0 = w_1 w_2 w_1 \ldots = w_2 w_1 w_2 \ldots \) where products are reduced for certain \( \xi \in \mathbb{R} \)
and \( 2 \ell (w_0) = |S_1| \cdot h. \)

Assuming claim, if \( h \) even, then \( w_0 = (w_1 w_2)^{h/2} = \frac{w_c}{m} \) reduced.

so it is equality in braid gp.

if \( h \) odd, then \( w_0 = w_1 w_2 w_1 \ldots \) so this holds in braid gp.

now in braid gp: \( r(w_0)^2 = (w_1 w_2 \ldots w_1 \ldots ) = (w_1 w_2) \)
\( 2h \) times.

So one particular Coxeter elt.

satisfies this identity.

All Coxeter elements are in same cyclic shift class, so strongly conj, so conj in.

braid gp. so have relation like \( w : \) Coxeter \( w/ g w g^{-1} = W_0 \) with \( g W g^{-1} = r(w_0)^2 \)
so \( w h = r(w_0)^2 \).
so now we have $T_{W_0} = T_{W_e}$ in Hecke algebra.

If $\lambda_1, \ldots, \lambda_m$ are e-values of $T_{W_e}$ in rep $\rho$ in $\text{char.}$ irreducible, then $T_{W_0}$ has eigenvalues $\lambda^h_1, \ldots, \lambda^h_m$, and they all satisfy:

$$\lambda^h_i = \prod_{s \in S'} q_s^{f_s/h}$$

so $\lambda_i = \frac{\xi_i}{\prod_{s \in S'} q_s}$

$d^\text{th}$ root of unity

$\forall i = 1, \ldots, m.$

so $\chi(T_{W_0}) = \left( \sum_{i=1}^m \xi_i \right) \cdot \prod_{s \in S'} q_s^{f_s/h}$

Under specialization, $\xi_s \rightarrow 1$.

so can rewrite as

$$\chi(T_{W_0}) = \chi_1(w) \cdot \prod_{s \in S'} q_s^{f_s/h}$$

If $\chi = \chi^{(d)}_v$ : character of $d^\text{th}$ exterior power of reflection rep $\rho$.

the above reads: $\chi^{(d)}_v(T_{W_0}) = \chi^{(d)}_{V_{11}}(w) \cdot \prod_{s \in S'} q_s^{f_s/h}$

claim:

(1) $\chi^{(d)}_{V_{11}}(w[n]) = (-1)^d$

For $S_n$:

(2) Setting $q_s = q$ \quad $\forall s \in S'$,

$$\sum_{s \in S'} f_s = h \cdot (|s| - d) \quad \text{for} \ \chi^{(d)}_v$$

(3) If $\chi$ irreducible, $\chi \neq \chi^{(d)}_v$ some $d$, then $\chi(T_{W[n]}) = 0$
To do general case, we claim that our identity for $\lambda = [n]$
extends to direct products of Symmetric grps -

If given Young subgp. $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_r}$, $w_\lambda$: product of
Coxeter elements
reflection repn is $V_{W_\lambda} := V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r}$.

so calculating on each block gives result:

determinants multiply, traces add

so $\psi(Tw_\lambda) = \frac{1}{|S_\lambda|} \sum_{w \in S_\lambda} \psi_1(w) \det(q \cdot \text{id}_{V_\lambda} - \rho_\lambda(w))$

(14)

Now $Res_{KH}^{K_{H_\lambda}}(\chi(Tw_\lambda)) = \sum_{\psi \in \text{Irr}(K_{H_\lambda})} m(\chi, \psi) \psi(Tw_\lambda)$, where $m(\chi, \psi)$
is multiplicity of $\psi$
in $Res^{KH}_{KH_\lambda}(\chi)$

Of course, character value of restricted repn
is same as that of $\chi(Tw_\lambda)$ on $KH$.

Substituting (14) we obtain double sum and we can reverse order of
summation:

$\chi_1(w) = \frac{1}{|S_\lambda|} \sum_{w \in S_\lambda} \left( \sum_{\psi \in \text{Irr}(K_{H_\lambda})} m(\chi, \psi) \psi_1(w) \right) \det(q \cdot \text{id}_{V_\lambda} - \rho_\lambda(w))$

and now notice summands depend only
on conj. class in $S_n$, not $S_\lambda$.

(As $\rho_\lambda(w)$, $\rho_\lambda(w')$ conj. linear transfs if $w \sim w'$)