Algebraic gps. — Study matrix gps. like $GL_n(F), F$ field.

If $F = \mathbb{R}, \mathbb{C}$ then has structure of Lie gp. (group and diff manifold)

We won't $F = GF_q$, finite field or $\mathbb{Q}_p$: $p$-adic completion. Need algebraic groups.

For us, need only affine varieties.

Over algebraically closed fields, particularly simple: $K$

$V$: subsets of $K^n$ which are simultaneous zeros of $\{f_i\}_{i \in S}$

variety $\phi \quad \eta \rightarrow \mathfrak{m}$ ideals: $f \text{ s.t. } f(v) = 0 \forall v \in V$

vanishing set $\eta \quad \phi \rightarrow \mathfrak{m}$ define ideal.

and $V(\mathfrak{m}) = V$ but $J(V(\mathfrak{m})) = \sqrt{\mathfrak{m}}$

affine subvarieties of $V$ form closed sets

in topology — Zariski topology.

Algebraic gp. $/K$ is a set $G$ which is an algebraic variety $/K$

and group with group maps in right category:

$$(x, y) \mapsto xy \quad x \mapsto x^{-1}$$

are morphisms of varieties.

Really a functor of all commutative $K$-algebras $E$.

giving gp. structure to rational points $G(E)$.

\[ \text{in bijection with algebra homoms. on coordinate rings} \]

\[ K[G] = K[x_1, \ldots, x_n]/J(V). \]

(maps continuous in Zariski topology.

In affine entry, these arise as polynomial maps restricted from $K^n \rightarrow K^n$ affine spaces.)
Example: \( GL_n(K) = \{ (a_{ij}) \in K^{n \times n} \mid \det(a_{ij}) \neq 0 \} \)

Problem: not closed subset of \( K^{n \times n} \), not affine variety.

\[ = \{ (a_{ij}), b) \in K^{n \times n+1} \mid b \cdot \det(a_{ij}) = 1 \} \]

easy check that operations mult and inverse are poly. expressions \( \checkmark \) in \( (a_{ij}), b \).

so closed subgps of \( GL_n(K) \) are also affine algs. gps. "linear algebraic gps"

And in fact every affine algs gp \( \cong GL_n(K) \) for some \( n \).

other examples: In \( GL_2(K) : \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \lambda \in K.

multiplication = addition in entry 1,2.

\[ \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 + \lambda_2 \\ 0 & 1 \end{pmatrix} \]

call this isom. class \( G_a \) a "additive."

Whole gp. \( GL_1(K) \cong K^\times \), call this isom. class \( G_m \) "multiplicative"

both gps of dimension 1

where \( \dim(v) = \text{transc. deg. of Frac}(K[v]) / K \).

What is its Lie algebra? What about lie group case?

- Space of left-invariant vector fields

- Elements whose exponential map lives in lie gp along one-parameter subgp. (matrix groups)

- tangent space for identity.

this we can adapt, using correspondence between \( V \) and its

coordinate ring \( K[v] = K[x_1, \ldots, x_m] / \mathfrak{m}(v) \).
Have derivations $D : K[V] \to K_v$ linear map s.t. $(K_v : K$ as $K[x]$ module by mult. by $f(x))$

$$D(fg) = Df \cdot g(v) + f(v) \cdot Dg \quad \forall f, g \in K[V].$$

$T_v(V) : \text{ set of all such derivations.}$

$$= \{ D_a = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n} \mid D_a (J(v)) = J(\{v\}) \}$$

- Why does it have Lie alg. structure? — (when $V \leftrightarrow G : \text{alg. gp.}$)

If $G$ algebraic gp. with coord. ring $K[G]$, define right mult. action by $x \in G$:

$$f^x(t) = f(t \cdot x).$$

Then $\alpha_x : f \mapsto f^x$ is $K$-algebra autom. of $K[G].$

So have homom $G \to \text{ Aut } (K[G])$

$\alpha_x \mapsto \alpha_x.$

Der $(K[G]) = \{ D : K[G] \to K[G] \mid D(f_1 f_2) = Df_1 \cdot f_2 + f_1 \cdot Df_2 \}$

$\forall f_1, f_2 \in K[G].$

Form Lie algebra with bracket

$$[D_1, D_2] = D_1 D_2 - D_2 D_1.$$

Check that it is bilinear, skew-symm. and satisfies Jacobid identity $= -[D_2, D_1]$.

Has subalgebra

Der $(K[G])^G : D \text{ s.t. } D \alpha_x = \alpha_x D.$

And $\text{Der } (K[G])^G \sim_{y.s.} T_e (G)$

$$D \mapsto (f \mapsto Df(e))$$

Example: $\text{GL}_n(K)$ is Zariski open set in $\text{End } (K^n)$ with tangent space $\text{ at } e = \text{ ambient space }$, so has Lie algebra $\text{ End } (K^n) = \text{Mat}_n(K) = g\text{ln}_n(K).$
Adjectives: \( G \) linear alg. group \( \subseteq GL_n(K) \).

\( T \in GL_n(K) \) is semisimple if diagonalizable.

\( T \in GL_n(K) \) is unipotent if all eigenvalues of \( T \) are 1.

(independent of embedding, so makes sense for \( G \).

a.k.a. fastful repn.

Jordan decomposition: Each \( x \in G \) uniquely expressed as

\( x = x_s x_u = x_u x_s \) \( x_s \) semisimple, \( x_u \) unipotent.

(homoms. of LAGs preserve semisimple, unipotent parts)

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Work over any field - just need to replace "algebraic variety" with proper definition over that field. \( F \)

then semisimple means diagonalizable over \( \overline{F} \): algebraic closure of \( F \)

the definition of unipotent oh.

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A LAG is said to be unipotent if all its elements are unipotent.

We say \( G \) is reductive if unip. radical is trivial.

Given an arbitrary \( G \),

\( \) unipotent groups are not nilpotent, but they are solvable:

The series

\[ G = G^{(0)} \supseteq G^{(1)} \supseteq \ldots \]

with \( G^{(i+1)} = [G^{(i)}, G^{(i)}] \) terminates in identity.

Also define radical of \( G \) as maximal normal solvable subgroup, \( G \) semisimple if radical is \( \{e\} \)

For \( GL_n(K) \), radical is scalar matrices as \( 1 \to K^* \to GL_n(K) \to \text{Pol}_n(K) \to 1 \)

and \( \text{Pol}_n(K) \) is simple. None of these are unip. except \( G \), so \( GL_n(K) \)

is reductive.