Given reductive alg. gp. \( G \) with maximal torus \( T \),

form characters \( X = \text{Hom}(T, \mathbb{G}_m) \), cocharacters \( Y = \text{Hom}(\mathbb{G}_m, T) \)
in duality via non-deg. pairing \((\chi, \gamma) \mapsto \chi \circ \gamma\) as elt. of \( \mathbb{Z} \).

Given minimal proper subgps \( U_d \) of \( U, U^- \) normalized by \( T \),

\[ \forall d \in X = \text{Hom}(T, \mathbb{G}_m) \quad \text{from action of } T \text{ on } U_d \cong \mathbb{G}_a \]

by conjugation.

\( d \) "root" of \( G \), \( \Phi \); set of all roots.

\( \Phi^- \): roots in \( U^- \), \( \Phi^+ \): roots in \( U \)

**Fact:** \( G \) is generated by \( T, U_d : d \in \Phi \).

In \( GL_n \), \( U_d \)'s are of form \( U_{d_{ij}} = \{ I_n + \mu \cdot E_{ij} \mid \mu \in K \} \)

with \( d_{ij} : (x_1, \ldots, x_n) \mapsto x_i x_j^{-1} \). Pick off exponents

\((0, 0, 1, 0, 0, -1, 0, 0)\) \( \uparrow \) \( T \) \( \uparrow \) \( i \) \( \uparrow j \)
E_1 in GL_n, U_{d_1}'s are given by one-parameter subgps
so label them U_{d_1} according ly.
\[ \xi \in \mathbb{Z} \Rightarrow E_{ij} \quad (\xi \in \mathbb{Z}) \]
\[ \text{elem. matrix with } 1 \text{ in position } (i,j) \]

roots \( \alpha_{ij} \) are elements of
\[ \text{Hom} \left( T, G_m \right) \text{ corresponding to cong action} \]
\[ t = \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) \quad U_{ij} = \left( \begin{array}{c} 1 \\ \vdots \\ \mu \end{array} \right) \rightarrow \text{ then cong multiplies} \mu \text{ by } x_i x_j^{\text{col j}} \]
so \( \alpha_{ij} : \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) \rightarrow x_i x_j^{\text{col j}} \)

Good way to think about reductive gps - built out of embedded copies of \( SL_2 \)

If \( \alpha \) root, \( -\alpha \) opposite root
( in corresponding \( U^- \) to \( U \)
related by conjugation )
of corresp. \( B, B^- \)

Consider \( \left< U_{\alpha}, U_{-\alpha} \right> \subseteq G \). Turns out this is either \( \simeq \text{ SL}_2(K) \)
or \( \text{ PGL}_2(K) = \text{ GL}_2(K)/\pm 1 \)

and \( \exists \) homom. \( \phi_{\alpha} : SL_2(K) \rightarrow \left< U_{\alpha}, U_{-\alpha} \right> \)
s.t. \( \phi_{\alpha} \left( \begin{array}{cc} 1 & * \\ 0 & 1 \end{array} \right) \rightarrow U_{\alpha}(*) \)

\[ \phi_{\alpha} \left( \begin{array}{cc} 1 & * \\ 0 & 1 \end{array} \right) \rightarrow U_{-\alpha}(*) \]

then \( \phi_{\alpha} (\lambda, \chi) \) is 1-dim'l subgp. of \( T \). Set
\[ \alpha^\vee = G_m \rightarrow T \]
\[ \lambda \mapsto \phi_{\alpha}(\lambda, \chi) \quad \left< \alpha, \alpha^\vee \right> = 2. \]
For example in $\text{GL}_n(K)$:

$$\alpha_{ij} : \lambda \mapsto \left( \begin{array}{l}
\lambda_1 \\
\vdots \\
\lambda_i \\
\lambda_{i+1} \\
\vdots \\
\lambda_n
\end{array} \right)$$

We obtain elements

$$w_\alpha := \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \langle X_\alpha, X_{-\alpha} \rangle \quad \text{in} \quad N(C) , \quad \text{and one}
$$

may check that it acts on roots, coroots via:

$$w_\alpha (\alpha) = \alpha - \langle \alpha, \alpha^\vee \rangle \alpha^\vee$$

$$w_\alpha (\alpha^\vee) = \alpha^\vee - \langle \alpha, \alpha \rangle \alpha$$

and thus $w_{\alpha} = w_{-\alpha}$ and $w_{\alpha}^2 = 1$.

Fact: $w_{\alpha}$ generate $W := N(C)/T$.

In $\text{GL}_n(K)$, $w_{\alpha_{ij}}$ acts as transposition $(i \, j)$, generating $S_n = A_{n-1}$.

Classification Thm. for reductive gps: The quadruple $(\alpha, \alpha^\vee, \beta, \beta^\vee)$

is called a "root datum".

Given $G \cong (\alpha, \alpha^\vee, \beta, \beta^\vee)$

in reverse direction, if $\alpha, \beta$ free abelian gps of equal rank

with non-deg pairing $\alpha \times \beta \rightarrow \mathbb{Z} = \langle, \rangle$

$\beta, \beta^\vee$ finite sets with associated reflections $w_d : \beta \rightarrow \beta$ preserving $\beta, \beta^\vee$ in bijection with $\langle \alpha, \alpha^\vee \rangle = 2$ reduced crystallographic root systems (finds subset closed under reflections $S_d$)

then $\exists \, G/\mathbb{K} = K$ closed with this root datum.
So given \( G \) as root datum \((X, \Pi, Y, \Pi')\) with Weyl group \( W = \langle \omega_d \rangle \) \(\omega_d \in \Pi\). 

\( W \) is a Coxeter gp. Which Coxeter gps can arise?

- Can prove \( \Pi \) is reduced (so assoc. Cartan matrix has \( c_{s,t} = c_{t,s} \) if \( m_{s,t} \) odd).
- (only mult. of \( d \) in \( \Pi \)
  
  are \( \pm \alpha \))

\( \Rightarrow \) \( m_{s,t} = 2, 3, 4, \) or \( 6 \).

(Other values result in \( c_{s,t} \)'s not rational.)

For Finite Coxeter gps: Families \( A, B, D, \) dihedral, \( + \) finite list.

- 

  \( A, B, D \) as before (all roots have same length.)

  \( B \) Coxeter gp has two forms. Classes of root systems

\( \rightarrow \) 

Coxeter gp with \( \alpha_6 \) is also ok.

\( G_2 \) \( \alpha_6 \)

\( F_4 \) \( \alpha_6 \)

\( E_6, E_7, E_8 \) simply laced

H's not possible - have order 5 relations...

Q: Do there exist groups \( G \) attaining these? Is this a classification of reductive alg. gps?

Pf of (x) - or more refined statement:

\[ W \delta \cdot \chi = \chi - \langle \chi, \alpha_d \rangle \alpha \quad \forall \chi \in X \]

integer valued

\( \chi \Pi \rightarrow \mathbb{Z} \)

Reductive gps have coroot lattice in duality with root lattice by integral form. Not possible for all finite Coxeter gps.
Of course, root systems alone won't determine root datum.

Examples: \( \text{GL}_n, \text{SL}_n, \text{PGL}_n \) all have root system \( \text{An-1} \cong \text{Sn} \).

\( \text{GL}_n \) has character lattice \( \Z^n \), \( \text{SL}_n, \text{PGL}_n \) have lattice \( \Z^{n-1} \).

(\( G \) is semisimple \( \Rightarrow \) char. lattice has same rank as root system)

But known even \( \text{SL}_2, \text{PGL}_2 \) not isomorphic: \( \text{SL}_2 \) contains \( \pm 1 \dagger \), \( \text{PGL}_2 \) has only one class.

Compute coroots for the two gps:

\( \text{SL}_2: \quad \chi = \langle \chi \rangle \quad \chi: (t^a t^b) \rightarrow t^a \)

\( \alpha: (t^a t^b) \rightarrow t^a \quad \Z\alpha = \langle 2\alpha \rangle \)

\( \alpha^\vee: t \rightarrow (t^a t^b) \quad \text{so } \gamma = \Z\alpha^\vee \).

\( \text{PGL}_2: \quad \text{Torus is } \langle \langle \begin{pmatrix} t^1 & t^2 \\ 1 & 1 \end{pmatrix} \rangle \rangle \quad \text{so has representatives} \)

\( \langle \begin{pmatrix} t^1 & t^2 \\ 1 & 1 \end{pmatrix} \rangle \)

\( \langle \begin{pmatrix} t & 1 \\ 1 & t \end{pmatrix} \rangle \)

\( \langle \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \rangle \)

\( \langle \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \rangle \)

\( \langle \begin{pmatrix} 1 & t \\ t^2 & 1 \end{pmatrix} \rangle \quad \text{algebraic characters: } (t^a, t^b) \rightarrow t^a \quad \text{so } \langle \beta \rangle \text{ with } \beta \text{ corresp. to } n=1. \)

\( \chi \) and \( \beta \) is a root.

while \( \beta^\vee: t \rightarrow (t^a t^b) \quad \text{according to } \langle \beta, \beta^\vee \rangle = 2. \)

so \( \Z\beta^\vee = 2\gamma \), \( \gamma \) cocharacter (lattice).

Upshot: In \( \text{SL}_2 \) and \( \text{PGL}_2 \) roles of \( \chi, \gamma \) with \( \beta \) and \( \beta^\vee \) are reversed.

not the same root datum.