We were asserting that smooth $G$-modules $\leftrightarrow$ smooth $H(G)$-modules

$$V = \bigcup_{k} V^k \quad \quad H(G) \ast M = M$$

and we proved that $H(G) \otimes_{H(G)} M \rightarrow M \quad \text{i: induction.}$

with left-hand side having smooth $G$-module structure, given by

left translation by $g^{-1}$.

We wish to check that $r \circ i = \text{id}$,

where \( r : (\pi, \nu) \rightarrow H(G) \text{ module with action } \pi(f \cdot \nu) = \int_{G} f(g) \pi(g) \nu \) \( G \quad dg(g) \)

i.e. check $\pi(f) \cdot m = f * m$

if $\pi$ is repn constructed above by induction.
Pick an $M$, choose open compact s.t. $K$ of $G$ s.t.

$$e_K * m = m.$$  

Then $e_K \otimes m \mapsto e_K * m = m$ in isomorphism.

Acting by left translation, to obtain $G$-action,

$$\pi(g), m = \frac{1}{\mu(k)} l_g k * m. \ (*)$$

Two things to check:

1. This $G$-module action maps to $H(G)$ action we started with:

   $$f \mapsto \pi(f) \text{ agrees with } f * m.$$

Recall that

$$\pi(f) \cdot m = \mu(k) \sum_{g \in G/k} f(g) \pi(g) \cdot m$$

where $K$ open compact s.t.

- fixing $m$
- and $f$ under rt. translation

( if we choose $K$ small enough, then condition satisfied and

$$\pi(f) \cdot m = \sum_{g \in G/k} f(g) l_g k * m$$

$$f = \sum_{i=1}^r c_i l_{g_i} k$$

for $g_1, \ldots, g_r$ reps for $G/k$. )
more of comment than something to check:

(*) implies that any \( H(G) \) homom. is a \( G \)-homom. of associated representations. \( \varphi : M \to M' \quad H(G) \)-homom.

then \( \varphi(f \ast m) = f \ast \varphi(m) \)

\[ \varphi(\pi(g) \cdot m) = \pi(g) \cdot \varphi(m) \checkmark \]

So "smooth repn of \( G \)" \( \leftrightarrow \) "smooth \( H(G) \)-module"

Just as before, use \( \pi(\epsilon_k) \) to project \( V \to V^k \) to obtain \( H(G/K) \)-module on which \( \epsilon_k \) acts as identity.

Proposition: If \( V \) irreducible, then \( V^k = \mathfrak{0} \) or irreducible \( H(G/K) \)-module.

and we have bijection between \( \begin{array}{c} \{ \text{smooth irreps of } \quad G \text{ with } \quad V^k \neq 0 \} \\
\sim \end{array} \)

\( \begin{array}{c} \{ \text{smooth } H(G/K) \text{ modules} \} \\
\sim \end{array} \)

if almost identical to the finite gp case
proof using idempotents. — irreducibility of \( V^k \) is same.

In induction map \( i : \quad M \longrightarrow U := H(G) \otimes H(G/K) \)

then \( U^k \cong M \) — same as before: \( U^k = \epsilon_k \ast H(G) \otimes M \)

\( H(G/K) \)-mod

\( \epsilon_k \otimes M \cong M \).

Trouble: \( U \) may not be irreducible.

Need to cut out subspace \( X \) on which \( X^k = 0 \).
By Zorn's lemma, if $G$-subspace $X$ of $U$, maximal with respect to the property that $X \cap e_k \otimes M = 0$ (i.e., $X^k = 0$),

Given any other $G$-subspace $Y$ of $U$ with $Y^k = 0$, then

$$(X+Y)^k = X^k + Y^k = 0$$

so must have $Y \subseteq X$ by maximality of $X$. Hence $X$ is unique maximal such $G$-subspace.

**Claim:** $U/X = : V$ is irreducible $G$-module.

Indeed if $W \not\subseteq X$, then $W \cap e_k \otimes M \neq 0$, so $W$ must contain the simple $H(G/K)$-mod. $e_k \otimes M$ so $W = U$,

and $V^k = M$ as $H(G/K)$-module.

Uniqueness proved similar to method before.

(extend any $H(G/K)$-hom. $f: M \to M'$ to any $H(G)$-hom $f: H(G) \otimes H(G/K) M \to H(G) \otimes H(G/K) M'$)

**Claim:** $f(X) = : X'$ is unique maximal $H(G)$-subspace of $U'$ s.t. $(X')^k = 0$,

so $f$ induces isom. $U/X \cong U'/X'$.
In number theory, we study subgroups of $SL(2, \mathbb{Z})$ classically in theory of modular forms.

With host of finite index subgroups determined by

Congruence conditions $\Gamma(n)$ ( $\equiv I_2 \mod n$ )

$\Gamma_0(n)$ ( $(a \ b), c \equiv 0 \mod n$ )

Weigh the benefit - detecting more reps, with cost - dealing with higher index $\Gamma_1(n)$ ( $(a \ b), a, d \equiv 1 \mod n$, $c \equiv 0 \mod n$ )

Similar analogies in $p$-adic world: (say $SL(2, \mathbb{Q}_p)$ or $GL(2, \mathbb{Q}_p)$ or another red. gp.) $\Gamma(n)$ replaced by open compact subgroups $I_{n+} \cdot \mathfrak{p}^\infty \cdot \text{Mat}_n(\mathbb{O})$

$\Gamma_0(n)$ replaced by "Iwahori subgroup" and its higher order analogues.

( $\equiv I \mod \mathfrak{p}^n$ )

(and $SL(2, \mathbb{Z})$ itself $\hookrightarrow GL(2, \mathbb{O})$, or $G(\mathbb{O})$ in general)

Take $G(\mathbb{O}) \to G(\mathbb{O}/\mathfrak{p})$

If $\mathbb{F}_q$ finite field with $q$ elements, e.g. $\mathbb{F}_p$ for $\mathbb{Q}_p$

and consider inverse image under $B(\mathbb{F}_q)$. This is "Iwahori" subgroup.

E.g. for $GL_n(\mathbb{Q}_p)$:

\[
\begin{pmatrix}
0 & & \\
& \ddots & \\
& & 0
\end{pmatrix}
\begin{pmatrix}
\mathfrak{p}^n & \\
& \mathfrak{p}^n
\end{pmatrix}
\begin{pmatrix}
\mathfrak{p}^n & \\
& \mathfrak{p}^n
\end{pmatrix}
\]

Lower triangular entries divisible by $\mathfrak{p}$.

Big question: What is structure of $H(G/K)$ for various choices of $K$?

Maximal compact e.g. $G(\mathbb{O}) = K$. So most natural first choice.

In automorphic forms, each form has local constituents which are $G(\mathbb{O})$ fixed at almost all places.

What is a set of coset representatives for $K^G/K$?

Reminds us of Cartan decomposition in real or Lie gps:
p-adic Cartan decomps. (e.g. for \( G = \text{GL}_n \)):

Every double coset \( K \backslash G(F)/K \), \( K = G(\Theta) \), has unique representative of the form
\[
\begin{pmatrix}
\lambda_1 & \cdots & \lambda_n \\
\cdots & \\
\cdots & \\
\lambda_n & \\
\end{pmatrix}
\]
\( \lambda_1 \gg \lambda_2 \gg \cdots \gg \lambda_n \).

In general, these elements are in \( T(F)/T(\Theta) \), \( T(\Theta) = G(\Theta) \cap T(F) \)

and we have a \( T \)-action on \( T(F) \) ordering them in this way.

- Hecke algebra is commutative.
  
  irreducible:
  
  Since smooth reps with \( \leftarrow \) simple \( k \)-fixed vectors \( H(G//K) \)-modules, then if \( V \) admissible (so \( V^k \) finite-dimensional)

  \( V^k \) is at most one-dimensional.

  (sketch for \( G = \text{GL}_n \)) \( i : H(G//K) \to H(G//K) \)
  
  "Gelfand Trick" \( i(f(x)) = f(Tx) \) \( T \) : transpose.

  Since \( G(\Theta) \) stable under transposition, this map is well-defined.

  Transposition reverses order of multiplication, so \( i(f_1 * f_2) = i(f_2) * i(f_1) \)

  But \( i \) is also the identity map on double cosets, since they have reps that are diagonal ds,

  conclusion: identity map is anti-automorphism

  so \( i(f_1 * f_2) = \text{id} * i(f_1) * i(f_2) \)

  = anti-autom \( i(f_2) * i(f_1) \)
Additional notes on Cartan decomposition over p-adic field.

\[ T(F) = \text{split, maximal torus} \cong (F^*)^r \]

\[ T(F)/T(0) = \left\{ (\omega^{m_1}, \ldots, \omega^{m_r}) \mid m_i \in \mathbb{Z} \right\} \cong \mathbb{Z}^r \]

must be units

picturing \( \text{GL}_r \) as example:

\[
\begin{pmatrix}
\omega^{m_1} \\
\vdots \\
\omega^{m_r}
\end{pmatrix}
\]

with diagonal matrices.

think of this as map \( \omega \mapsto \begin{pmatrix} \omega^{m_1} \\ \vdots \\ \omega^{m_r} \end{pmatrix} \)

and extend to cocharacter, in \( X_*(T) \cong \mathbb{Z}^r \)

called this \( \gamma \) in our root datum disc.

write \( \omega^M = (\omega^{m_1}, \ldots, \omega^{m_r}) \)

for element in \( T(F) \hookrightarrow \text{cocharacter} \mu \).

\( W \)-action on \( T(F) \), ask for unique element in any \( W \)-orbit.

i.e. on \( X_*(T) \) get dominant cocharacters by asking for \( \mu \)

\[ \mu^w(t) = \mu(t)^w \]

s.t. \( \langle \alpha, \mu \rangle \geq 0 \) \& \( \alpha \) : simple roots.

We can identify coweight vector space with that of root vector space by

setting \( \alpha^\vee = 2\alpha \) with \( \alpha \in \Phi \).

\[ \langle \alpha_1, \alpha \rangle \]

co-weight lattice abstractly:

\[ \mathfrak{p}^\vee := \left\{ \lambda^\vee \in V^\vee \mid \langle \alpha, \lambda \rangle \in \mathbb{Z} \quad \forall \alpha \in \Phi \right\} \]
Example of repin with $K$-fixed vector, $K = G(\Theta)$

Want to do Harish-Chandra induction. Start with char. $X : T(F) \to \mathbb{C}^*$

Want $K$-fixed vector, so really want $X : T(F)/T(\Theta) \to \mathbb{C}^*$

then inflate to $B(F)$, induce to $G(F)$.

Form: $\text{Ind}_{B}^{G}(X) = \int \left\{ f : G \to \mathbb{C} \mid f(bg) = X(b)f(g) \right\}$

Yet another decomposition theorem: $G = B \cdot K$ $K = G(\Theta)$, $B$: Borel.

with $B \cap K$ s.t. $X(b) = 1$ if $b \in B \cap K$.

so $f_{\gamma} \equiv X(b)$ if $\gamma = bk$

Span 1-dim'l space of $K$-fixed vectors.

In practice, nicer to add $\delta^{1/2}(b)$

where $d_K(b) = \delta(b) d_L(b)$

where $d_K, d_L$ are right, left Haar measures

on $B$; non-reductive $G_F$.

e.g. $\Theta_{L-n} : \delta \begin{pmatrix} x_1 & \cdots & x_n \\ 0 & \ddots & \vdots \\ 0 & \cdots & x_n \end{pmatrix} = \prod x_i^{n-1} \cdots 1_i x_i^{1-n}$

where $x_i^{1-3}$