Last time, we had shown
\[ \text{Waff} = \langle S, \mu \rangle \cong \mathbb{Z}^2 \times \mathbb{W} \]

if \( G \) not simply connected, replace this with \( \mathbb{T}(\mathbb{F})/\mathbb{T}(0) \)

Wanted to show Waff is a Coxeter gp.

For generating set \( S = \) walls of fundamental alcove \( \alpha \in \mathbb{H}^+ \)

\[ \alpha \leq \mathbb{H}^+ \] highest root

End of last class: \( \text{Waff} \) acts transitively on alcoves, or rather \( \langle S \rangle \leq \text{Waff} \) acts transitively

(acts discretely on \( V_{//} = \mathbb{Z}^2 \oplus \mathbb{R} \))

Didn't get to say that \( \langle S \rangle = \text{Waff} \).

Given \( s_{d,1,k} \in \text{Waff} \), pick alcove \( A' \) with \( s_{d,1,k} \) as wall.

By transitivity of action, \( \exists \langle S \rangle \) s.t. \( W \cdot A' = A \), the fundamental alcove.

Then \( W \cdot P_{d,1,k} \) is a wall of \( A \), i.e. \( W s_{d,1,k} W^{-1} = S \), \( s \): simple reflection.

But then \( s_{d,1,k} = W^{-1} s W \in \langle S \rangle \). \( \forall s \in \langle S \rangle \)

\( j \in \{ 0, \ldots, r \} \)
Now that we know $W_{aff} = \langle S \rangle$, define length of $w \cdot W_{aff}$ as shortest expression as a product of simple reflections. (no longer bounded function)

In finite case, had characterization $l(w) = \left| \exists x \in \mathbb{R}^+ \mid w(x) \in \mathbb{R}^- \right|$.

Want analogues of this in affine case:

Integer-valued function $n(w) = \left| \exists P_{aff} \text{ separating } A \text{ and } w \cdot A \right|$.

**Goal:** show $n(w) = l(w)$.

**Remark:** If we define "affine coroots" as linear functionals $L_{aff}: V \rightarrow \mathbb{R}$,

$$v \rightarrow \langle \alpha, v \rangle$$

then $L_{aff}$ vanishes on hyperplane $P_{aff}$ has $W$-action: $W \cdot L_{aff}(v) = L_{aff}(w^{-1}v)$.

and then $L$ is positive if its values are positive on $A$

negative \(\rightarrow\) negative

**Claim:** $n(w) = \left| \exists L_{aff} \text{ positive } \mid w \cdot L_{aff} \text{ negative } \right|$.

**Proof of Claim:** Given $L_{aff}$ positive, then $w \cdot L_{aff}$ negative means $L(w^{-1}v) < 0$ if $v \in A$. Hence $P_{aff}$ is hyperplane between $A$ and $w \cdot A$ (as its values are positive on $A$ and negative on $w \cdot A$).

Now we repeat proof technique for finite Coxeter.

Note $n(w) < \infty$ since any path between points in two alcoves meets finitely many of the parallel hyperplanes $P_{aff}$ for fixed $\alpha$. 

Write $\Pi_i$ for $i \in \{0, \ldots, n\}$ to denote $P_{i,0}$ and $P_{i,-1}$, with corresponding functions $L_i$.

Proposition: 
$$n(ws_i) = \begin{cases} n(w) + 1 & \text{if } w(L_i) \in \mathbb{R}^+ \setminus \mathbb{R}_{\text{aff}}^+ \\ n(w) - 1 & \text{if } w(L_i) \in \mathbb{R}_{\text{aff}}^- \end{cases}$$

Given we $w \in \mathbb{R}^+$, so
$$w = \vert \mathbb{R}^+_{\text{aff}} \cup w^{-1} \mathbb{R}^{-}_{\text{aff}} \vert$$

What is $x_i \mathbb{R}^+_{\text{aff}}$? 
$$N(s_i) = \Pi_i,$$

$w = \Pi_i$ since only a hyperplane separating $w \in \mathbb{R}^+$ and $w = s_i(L)$ negative implies $v, s_i v$

i.e. $s_i(L) \in \mathbb{R}^-_{\text{aff}} \iff L = L_i$.

So then $x_i \mathbb{R}^+_{\text{aff}}$ replaces $L_i$ in $\mathbb{R}^+_{\text{aff}}$ with its negative.

$$(-L_{d_1} k = L_{-d_1} k)$$

$$\Rightarrow n(ws_i) = n(w) + 1 \text{ if } L_i \in W^{-1} \mathbb{R}^+_{\text{aff}} \text{ and } = n(w) - 1 \text{ otherwise}.$$ 

Finally, we turn to prove by induction (just as in finite case)

Theorem: 
$$N(w) = \{ s_{i_1} \Pi_{i_1}, s_{i_2} \Pi_{i_2}, \ldots, s_{i_r} \Pi_{i_r} \}$$

if $w = s_{i_1} \cdots s_{i_r} \in \mathbb{R}^+_{\text{aff}}$

from which we conclude that $\mathbb{R}^+_{\text{aff}}$ acts simply: if $w A = A$ then $l(w) = 0$

i.e. $w \in \mathbb{R}^+$

and $l(w) = n(w)$. 
In our picture $wA = S_2 S_0 A$

and our path travels first to $S_2 A$ crossing $H_{d_{2,0}}$

and then $S_2 S_0 A$ crossing $H_{d_{1,1}} = S_2 H_{-d_{0,1}}$

$H_2$ for short

$H_0$ for short
Two proofs of Coxeter presentation -

(1) Use Exchange/Deletion condition

Exchange condition: We have with \( w = s_i \cdots s_r \) reduced. \( s_i \in S \),

Suppose \( l(ws) < l(w) \) for \( s \in S \). Then \( \exists j \) s.t. \( ws = s_i \cdots \hat{s}_j \cdots s_r \)

\[ \text{Proof: } N(w) = \text{Hyperplanes between } A_n, wA \]

\[ = \left\{ \mathbf{H}_{s_i}, s_i \mathbf{H}_{s_i}, \ldots, s_i \cdots s_{i+1} \mathbf{H}_{s_i} \right\} \text{ by earlier result.} \]

So \( N(w) = \left\{ \mathbf{H}_{s_i}, s_i \mathbf{H}_{s_i}, \ldots, s_i \cdots s_{i+1} \mathbf{H}_{s_i} \right\} \) (noting \( s_i H_{s_i} = H_{s_i} \))

Now by assumption, \( N(ws) < N(w) \) so \( w(L_s) \in F_{s_i} \)

(i.e. \( l(ws) < l(w) \)) \( \Rightarrow H_s \in N(w) \)

So \( H_s = s_i \cdots s_{i+1} H_{s_i} \) for some \( j \in [1,r] \).

Weyl group action on coroot letters corresponds to conjugation in Weyl group.

So \( s_i \cdots s_{i+1} s_i s_{i+1} \cdots s_r = w \)

\( \Rightarrow s_i \cdots s_r = s_{i+1} \cdots s_r \) \( \Rightarrow w = s_i \cdots s_j \cdots s_r \) \( \checkmark \)

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Exchange Condition \( \iff \) Deletion Condition \( \Rightarrow \) \((Waff, S)\) is a Coxeter gp.

Deletion Condition: \( w \) not reduced, \( \exists \) pair of indices \( i \neq j \) with

\[ W = s_i \cdots s_r = s_i \cdots \hat{s}_j \cdots s_r \] \( \text{non-reduced} \)

"Delete a pair without changing w"

\[ \wr \text{ In general, if } w \cdot \mathbf{H}_{d_{i,j}} = \mathbf{H}_{p_{i,j}}, \text{ then } w \cdot S_{d_{i,j}} = S_{p_{i,j}} \]

- analogue of \( w \cdot S_{d_{i,j}} = S_{w(s)} \) for finite gos.
prove Exchange ⇒ Deletion: Since \( l(m) < r \), \( \exists \) smallest \( k \) s.t.

\[ l(S_i \cdots S_{ik}) < k \] (not \( k+1 \)) so then \( l(S_i \cdots S_{ik}) = k-1 \)

and \( l(S_i \cdots S_{ik+1} \cdots) < l(S_i \cdots S_{ik}) \)

so exchange ⇒ \( \exists \) \( j \) s.t. \( S_i \cdots S_{ik-1}S_j = S_i \cdots S_j \cdots S_{ik-1} \), implying the result upon mult.
both sides by \( S_{ik+1} \cdots S_i \).

From deletion to Coxeter presentation.

Know Waff generated by \( S \), with simple reflections \( S_1, \ldots, S_r \)

st. \( S_i^2 = 1 \).

Plan: Show that all possible relns in \( S_i \)'s \( \text{ which take form} \)

\[ S_i \cdots S_j = 1 \] \( \text{ can be reduced to} \) \( (S_i S_j)^m = 1 \) \( \text{(Coxeter relations)} \)

pf. by induction on \( r \) (always even since \( \det(s_i) = -1 \))

Write \( k = 2g \). \( S_i \cdots S_{ik} = 1 \) ⇔ \( S_{i+1} \cdots S_{ik} = S_{i+1} \cdots S_{ik+1} \) \( \text{ (} \\text{only bad case is when } j = 1, k = 2g+1. \text{ Then exactly } r \text{ terms} \)

(\text{in relation. Can't assume it!})

\text{BAD: } S_{i+1} \cdots S_{ik+1} = S_{i+1} \cdots S_{ik} \text{ \( \star \star \text{BAD} \)}

\( \text{Repeat with } (\star) \text{ replaced by } S_{i+1}S_iS_{i+1} = 1, \text{ then} \)

\( \text{BAD: } S_{i+1} \cdots S_{ik+2} = S_{i+1} \cdots S_{ik+1} \text{ \( \star \star \text{BAD} \) )}

\( \iff S_i (S_{i+1} \cdots S_{ik+1}) S_{i+2} \cdots S_{ik+1} = 1 \text{ \( \star \star \text{BAD'} \) .} \)

\( \text{r terms. Try again as if it were } (\star) \)

\text{BAD: } S_{i+1} \cdots S_{ik+1} = S_{i+1} S_{i+2} \cdots S_{ik+1} \iff S_i = S_{i+1} \cdots S_{ik+1} \text{ \( \star \star \text{BAD' \) )}

Insert into \( (\star) \), done by induction.
For the second proof, note that relations in Waff are equivalent to having two paths from A to WA passing through different sequence of alcoves (and thus walls).

\[ A, s_i A, s_i s_{i+1} A, \ldots, s_i \cdots s_i A \text{ gives sequence of adjacent alcoves.} \]

To show Waff is a Coxeter gp, need to show any two such paths can be homotoped to one another using only \( s_i^2 = 1 \) and braid moves.

Choose paths to avoid affine subspaces resulting from intersection of hyperplanes in codimension 3. Homotopy can be done in complement, which is simply connected.

And we can pass through subspaces \( M \) of codimension 2 one at a time.

Take cross section perpendicular to \( M \) in \( V_{\mathbb{R}} \), project our path onto resulting plane, reduced to rank 2.

Handle rank 2 cases individually.

Bump's notes do case of type A.