unramified principal series:

\[ \chi : T(F) \to \mathbb{C}^* \], form repn \( \text{Ind}_{B}^{G} (\chi) \) (better:

\[ \text{Ind}_{B}^{G} (\delta_B \chi) \]

that is, view \( \chi \) as character of \( B \),
writing \( b = \varepsilon u, \varepsilon \in T(F), \ u \in U(F) \)

then setting \( \chi(b) = \chi(\varepsilon) \) "inflate to \( B \)"

If we take \( \chi \) trivial on \( T(\theta) \), so \( \chi : T(F)/T(\theta) \to \mathbb{C}^* \)

"unramified"

then this repn \( i(\chi) \) will have

\( G(\mathfrak{o}) \)-fixed vectors (Iwasawa decomposition \( G = BK \)

\[ B : \text{Borel } \quad K : \text{max'd comp.} \]

over local field \( G(F) = B(F) \cdot G(\mathfrak{o}) \)

with \( B(F) \cap G(\mathfrak{o}) = B(\theta) \).

\[ \chi \left( \pi^{m_1} \cdots \pi^{m_r} \right) = z_1^{m_1} \cdots z_r^{m_r} \]

for some choice of \( z_i \in \mathbb{C} \).

If \( z_i \)'s are in general position (ex. numbers for \( \delta_B \chi \) are not in same W-orbit)

then \( i(\chi) \) will be an irreducible representation,
(also smooth, admissible).

What is unique-up-to-scalar-mult. \( K \)-fixed vector? (\( K = G(\mathfrak{o}) \) here)

\[ f(g) = \delta_B^{-\frac{1}{2}} \chi(b) \chi(\varepsilon) \]

if \( g = \varepsilon b k : \text{Iwasawa decom.} \)

or further \( b = \varepsilon u, \varepsilon \in T(F)^*/T(\theta) \)

\[ f(b) \]

\[ = q^{ - \langle \rho, \mu \rangle} \]

\[ = \pi^{\mu} u \cdot \mu \cdot \pi^r \]

\[ \rho : \frac{1}{2} \sum_{\alpha \in \Phi^+} \]

Call \( f^0 \), where \( \circ \) is for "spherical"

\( K \)-fixed called spherical since \( \text{SO}(2,12) \)
is max'm comp of \( SL \).
Given \( i(x) \rightarrow i(x)^K = \langle f^0 \rangle \sim 1 \)-dimensional \( \mathcal{H}(G/k) \)-module, i.e. linear character of \( \mathcal{H}(G/k) \)

eigenvalue is in terms of \( \pi \), lets us recover principal series.

Automorphic form is vector in autom. repn of \( G(A_F) \) \( F \): Global field
\( A \): Adèlic ring
\( \pi = \bigotimes_v \pi_v \)

with \( f = \bigotimes_v f_v \).

For almost all \( v \), \( f_v \) is a \( K \)-fixed vector in unramified principal series.

To understand automorphic forms locally, almost all places, we can apply matrix coefficient. Try to build an \( L \)-function.

Example (Jacquet-Langlands 1970)
Global Whittaker function \( W(g, u) = \int_{F \backslash A} f(c(1, i, u)g, \alpha \mathbf{l}^{-1}) \psi(-u) \, du \)

\( \psi \): additive char. of \( A \) trivial on \( F \).
\( du \): gives \( F \backslash A \) measure 1.

Just like Fourier analysis, reconstuct \( f \) by ranging over various \( \psi' \) (all realized as \( \psi(m, u) \) for some \( m \in F^\times \),

by change of vars, move \( m \) to multiply \( g, u \).

Get \( f(g, u) = \sum_{m \in F^\times} W(g, \alpha(m, i) g, u) \)

Form \( L(s, f) = \int_{F^\times \backslash A^\times} f(c(1, i, u) \mathbf{l}^{-1}, \alpha) \left| \det A \right|^{s-1/2} \frac{d^* A}{\text{mult. Haar measure}} \)

Substitute Whittaker expansion collapse summation over \( F^\times \) with \( F^\times \) in quotient.
\[ = \int \sum_{m \in F^*} W(m \cdot 1) (a \cdot 1) \left| a \right|_{A^*}^{s-\frac{1}{2}} \, d^*a \]

\[ = \int W(a \cdot 1) \left| a \right|_{A^*}^{s-\frac{1}{2}} \, d^*a \]

\[ = \prod_{v \in \mathbb{A}^*} W_v(a_v \cdot 1) \left| a_v \right|_v^{s-\frac{1}{2}} \, d^*a_v \]

Want to explain reworking of results of Shintani- Kato- Casselman- Shalika —

who computed, in increasing levels of generality, the local Whittaker coefficients.

More algebra, less analysis:

Consider \( H(G/H) \)-module \( C_c \left( T(\mathbb{Q}) \backslash G / H \right) =: M \).

"universal" principal series.

Said that \( T(F)/T(\mathbb{Q}) \sim X_*(T) \) : character:

\[ R := \mathbb{C} [X_*(T)] \]

let \( \pi^A \mapsto \pi^A \in R \) be the "topological character" or "universal char."

then \( M \cong \text{Ind}_B^G \left( X_{\text{univ}}^{-1} J \right) \)

\[ \phi \in C_c \left( T(\mathbb{Q}) \backslash G \right) \mapsto \sum_{a \in T(F)/T(\mathbb{Q})} \delta_B^J(a \cdot a^{-1} g) \]

Way to study all characters \( \chi : T(F)/T(\mathbb{Q}) \rightarrow \mathbb{C}^\times \) at once. Indeed such a choice of \( \chi \) determines \( \mathbb{C}\)-alg. homom. \( R \rightarrow \mathbb{C} \), \( \Sigma \mapsto \Sigma \phi(z) \).