Discriminant of $\text{H} \gen W$ is a Laurent poly. in $q_i$'s.

There is a specialization homom. $k \to k^\times$ taking $q_i \mapsto q$

$\text{H} \gen W \to \text{H} q[W]$ and $\text{H} q[W]$ will be semisimple if $q \notin \{0, 1\}$ and $\{D(q) = 0\}$

finite set of points

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Final powerful result here:

Tits' Deformation Theorem: Before stating it, two definitions.

Separable algebra: semi-simple for every extension $L/K$ (including $K$ itself).

Let $A_L := A \otimes_K L$ is semisimple. (term comes from special case of $K[a_1, \ldots, a_n] = L$ with sq. minimal poly.) (if char. of $k = 0$, semi-simple.

this is immediate from trace criterion)

Numerical invariants: dimensions of simple modules for $A \otimes_K \overline{k}$

with $\overline{k}$: alg. closure of $k$. $A$ : separable algebra/$k$.

(there are $n_i$'s in Artin-Wedderburn $A \otimes_k \overline{k} = \text{Mat}_{n_i}(E) \times \cdots \times \text{Mat}_{n_r}(E)$)

Tits' deformation fam: $R$ integral domain, $\text{Frac}(R)$: field of fractions

$f: \begin{array}{c} \overline{k} \\to k \end{array}$ (specialization) hom. of rings

If $A$ is $R$-algebra with finite basis, and $A_f$ : algebra obtained by specialization.

If $A \otimes_R R^\times$ and $A_f$ are separable, THEN they have the same numerical invariants!
pf. of Tits' thm: \( T_k = \text{alg. cl. of } k = \text{Frac}(R), \ F = \text{alg. cl. of } F \)

\[ \dim_k A^k = n. \text{ Write } \text{"general elts" } a \in A^k = \sum_{i=1}^{n} a_i t^i \]

with associated characteristic polys.

\[ a_f \in A_f = \sum_{i=1}^{n} t^i a_i' \text{ where } a_i' = 1 \otimes a_i \]

(Pe k'[x], Pf \in F'[x], respectively.)

(Remember k', notation is \( k[t_1, \ldots, t_n] \), \( \text{Frac}(R[t_1, \ldots, t_n]) \).)

Extend \( f \) to polynomial rings \( f: R'[x] \rightarrow F'[x] \)

by simply mapping \( t_i \mapsto t_i' \). Then \( f(P(x)) = P_f(x) \)

(since structure const. in \( R'[x] \)

map to \( f(\text{str. const.}) \)

by previous lemma, (**) \( \overline{R} P_i(X)^{t_i} \) in \( \overline{F}'[x] \) with \( P_i : \text{numeralized invariants} \).

Just need analogous factorization of \( P_f(x) \) in \( \overline{F}'[x] \). For this,

need a couple standard facts from commutative algebra:

\( P \) is monic, so its roots are said to be "integral" over \( R' = R[t_1, \ldots, t_n] \)

and roots of \( P_i \) will be symmetric functions in these roots of \( P \),

so also integral over \( R' \), and live in \( \overline{E}' = \overline{E}[t_1, \ldots, t_n] \)

Fact: integral closure of \( R' \) in \( \overline{E}' \) is \( (R')' = \text{inf. cl. of } R \) in \( \overline{E} \)

(Prop. 13 in Ch. 5 of Bourbaki's Comm. Alg.)

So \( P_i(X) \in \overline{R}'[x] = \overline{E}[t_1, \ldots, t_n][x] \).
Second comm. alg. fact: \( f: R \to F \) extends to a homom. \( \bar{f}: \bar{R} \to \bar{F} \).

(and thus extension of \( f \) to \( R'[x] \to F'[x] \) extends to \( \bar{f}: \bar{R}'[x] \to \bar{F}'[x] \). )

Thus applying to (**) , \( \bar{f}(P(x)) = P_{\bar{f}}(x) \) since \( P \in R[x] \)

\[ = \prod_i \bar{f}(P_i(x))^{{}\overline{p}_i} \]

So we win if we can show that \( p_i \) are numerical invariants of \( A_f \).

This is immediate from (4) in previous lemma, since \( \deg (\bar{f}(P_i)) = \deg (P_i) = p_i \).

Corollary: \( H_q[W] \): Hecke algebra of fin. Coxeter gp \( W \) with \( q \in \mathbb{C} \)

\[ \cong C[W] \] provided \( q \) is not a root of \( \text{Disc} (H_q[W]) \)

(i.e. if \( H_q[W] \) is semisimple)

\[ H_{q_1}[W] \overset{f^1}{\cong} C[W] \] using \( \text{spec. } f^1: q \mapsto q_1 \).

\[ H_{q_2}[W] \overset{f_2}{\cong} H_q[W] \] using \( \text{spec. } f_2: q \mapsto q_2 \).

And \( H_{q_1}[W]/\text{Frac}(\mathbb{C}q_1) \) separable, \( \mathbb{C}[W], H_q[W] \) separable,

so all have same invariants by Titi's deformation theorem.

and this determines algebra/alg. closed field \( \mathbb{C} \) by Artin–Wedderburn.

Corollary of Tite's theorem: (Preserving notation so that $A^K, A^F$
Separable alg. $R$: int. closure of $R$ in $K$.
and $\overline{f}: \overline{R} \to \overline{F}$ extension of "spec" hom $f: R \to F$.)

$\mu$: irreducible character of $A^K$.

Then $\mu_{\overline{f}} : A^F \to F$ defines a character of $A^F$

\[ a_i \mapsto \overline{f}(\mu(a_i)) \]
(by extending linearly from map on basis $a_i, \overline{a_i}$)

and $\mu \leftrightarrow \mu_{\overline{f}}$ is a bijection from $\text{Irr}(A^K)$
to $\text{Irr}(A^F)$

If: Key lemma (3) gave every irreducible char. as

\[ \mu = \sum \xi_i a_i, \quad \mu \left( \sum \xi_i a_i \right) = \Psi_i(\mathbf{f}, \ldots, f_n) \quad \text{for any } \sum \xi_i a_i \in A^F \]

where this appeared as
next to highest coeff.
in irreducible factor $P_i(x)$
if $P(x)$.

Note that each of $P_i(x)$
were shown to be in $\overline{R}[X]$, so evaluating $\Psi_i$ at $(f_i, \ldots, f_n)$
gives elt. of $\overline{K}$, so can apply $\overline{f}$ to it.

Also showed $\overline{f}(P_i(x))$ is monic irreducible factor of $P_f(x)$, the char.
poly. of
thus again $-\overline{f}(\Psi_i) \in F'$ appears as next to
highest coeff. in factor $\overline{f}(P_i(x))$ and using (3) of
lemma, it again gives character.