Character theory for semisimple algebras (e.g., $\mathbb{C}[W]$, $H_\theta[W]$)

Given algebra $A$, module $V$, so have rep'n $T: A \to \text{End}(V)$
then the character of $V$ is just $\text{Tr}: a \mapsto \text{Tr}(T(a))$  
$A \to F$: underlying field.

Remember two representations are isomorphic (or "similar")

if given $T: A \to \text{End}(V)$, $S: A \to \text{End}(W)$  
f: $V \to W$  

such that $T(a) = f^{-1}S(a)f$  $\forall$ $a \in A$. Write $T \simeq S$ (or sometimes $V \simeq W$ if understood as modules)

First Fact: If $T,S$ rep's of semisimple alg. $A/F$, char$(F) = 0$,
then $T \simeq S \iff \text{Tr}(T(a)) = \text{Tr}(S(a))$  $\forall$ $a \in A$

"characters are equal"

$\text{If: (\Rightarrow)}$ if $T \simeq S$ then matrix $T(a)$ and matrix $S(a)$ are similar,
hence have same trace.

$\text{<=}$ if $\text{Tr}(T(a)) = \text{Tr}(S(a))$  $\forall$ $a \in A$,
write $A$ semisimple $= A_1 \times \cdots \times A_s$  
$A_i$: simple algebraic.

$\text{Mat}_{n_i}(D_i)$  
$p$-dimensional single simple module with single simple module $V_i$ with entries in $D_i$

(*) if $V,W$ are left $A$-mods, $f: V \to W$ homom of left $A$-mods

means $f(\lambda v) = \lambda f(v)$  $\forall$ $\lambda \in A$, $v \in V$.

For emphasis, write $f(\lambda T(a)v) = S(a) \cdot f(v)$

so if $f: \text{hom}$  
$V \to W$, $T(a)v = f^{-1}S(a) \cdot f(v)$
pf of \((\leq)\) in First Fact (continued):

consider \(e_i = (0, \ldots, 0, 1, 0, \ldots, 0)\) 1 : unit in \(A_i\)

if \(A = A_1 \times \cdots \times A_s\) then \(1\) in \(A = e_1 + e_2 + \cdots + e_s\)

and \(e_i\) acts on \(e_j\) of \(A\) as projection to \(i^{th}\) factor \(A_i\).

\(e_i^2 = e_i\) and \(e_i e_j = 0\) unless \(i = j\) (orthogonal idempotents)

Also \(a \cdot e_i = e_i \cdot a\) for all \(a \in A\), \(i \in [1, s]\) so \(e_i\)'s central.

Such an expression \(1 = e_1 + \cdots + e_s\) is called "central decomposition of identity"

( in fact there's a bijection between decompositions of \(A\) into direct products and central decomp. of identity)

Now write \(V \cong M_1 \oplus \cdots \oplus M_s\) with \(M_i : A_i\)-modules

\(\cong \bigoplus_{i \in I} V_i \cong \bigoplus_{i \in I} M_i V_i\)

\(\cong m_1 V_1 \oplus \cdots \oplus m_s V_s\)

similarly \(W \cong n_1 V_1 \oplus \cdots \oplus n_s V_s\).

We want to show \(m_j = n_j\) \(\forall j \in [1, s]\).

But \(\text{Tr}(T(e_i)) = m_i [V_i : \mathbb{K}]\) and \(\text{Tr}(S(e_i)) = n_i [V_i : \mathbb{K}]\).

By assumption, traces are equal, so \(m_i = n_i\) \(\forall i\) and hence

(in fact, proof shows only need traces = on center) \(V \cong W\).

Now let's construct some representations of Coxeter groups...
In $\mathbb{R}^2$, regular $m$-gon with a vertex at $(0,0)$ about origin

its group of rotations/reflects = \[ W = \bigoplus_{k} \text{clockwise rotations about angle } 2\pi k/m, \text{ } k \in [0,m), \]
m reflections in hyperplanes orthogonal to \[ e_k = (\cos \frac{k\pi}{m}, \sin \frac{k\pi}{m}) \]

order of $W = 2m$

\[ = \langle R : \text{rotation in } 2\pi/m, \text{ } S_0 : \text{reflection in hyperplane } \perp e_0 \rangle \]

as matrices

\[ R = \begin{bmatrix} \cos (2\pi k/m) & -\sin (2\pi k/m) \\ \sin (2\pi k/m) & \cos (2\pi k/m) \end{bmatrix}, \quad S_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \]

(acting on right $R \in R^2$)

Also present it as $\langle S_0, S_1 \rangle$ where $S_1 : e_{m-1} \mapsto -e_{m-1}$

in this basis $\{e_0, e_{m-1}\}$, the mats. are

\[ S_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 1 & 2\cos(\pi/m) \\ 2\cos(\pi/m) & 1 \end{bmatrix} \]

$S$: finite (index) set. Matrix $(C_{s,t}) = C$ with real entries

is a Cartan matrix if the following are satisfied:

1. if $s \neq t$ then $c_{s,t} \leq 0$, $c_{s,t} \neq 0 \iff c_{t,s} \neq 0$.

2. $c_{s,t} = 2 \forall s \in S, s \neq t$, \[ c_{s,t} c_{t,s} = 4 \cos^2 \left( \frac{\pi}{m_{st}} \right) \]
   with $m_{st} \in \mathbb{Z}_{\geq 2} \cup \{0\}$. 
Calculate \( 4 \cos^2 \left( \frac{\pi}{m_{s,t}} \right) \) with \( m_{s,t} = 2, 3, \ldots \)

<table>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>\ldots</th>
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<tbody>
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<td>1</td>
<td>2</td>
<td>\frac{3\sqrt{5}}{2}</td>
<td>\ldots</td>
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Ugly, can show with little trick that

\( 2, 3, 4, 6, \ldots \) only values for which

\( c_s, c_t, c_{s,t} \in \mathbb{Q} \).

Given Cartan matrix \( C = (c_{s,t})_{s,t \in S} \)

\( V \): vector space with basis \( \{ d_s \mid s \in S \} \)

\( GL(V) \): endom. of \( V \) (\( |S| \times |S| \) matrices with entries in \( \mathbb{R} \))

\( v, w \in V \) write \( v \circ w \).

Define action \( s: V \rightarrow V \)

\[ d_t \mapsto d_t - c_{s,t} d_s \quad (s, t \in S) \]

Easy lemma:

\[ d_s \cdot s = -d_s \quad (c_s, s = 2) \]

\[ \text{Tr}(s) = |S| - 2 \]

\[ s^2 = \text{id} \circ (V \circ V) \]

so \( s \) is diagonalizable autom. of \( V \) with \( |S|-1 \) e-values = +1

1 e-value = −1.

Finally, let \( W = \langle s \rangle \subset GL(V) \)

\[ \{ d_s \circ w \mid w \in W, s \in S \} = \Xi : \text{root system} \subset V \]. (invariant under \( W \) by def)

Thm: \( W \) has a presentation as Coxeter gp.:

\[ \langle s \in S \mid s^2 = 1, (st)^{m_{s,t}} = 1 \text{ if } m_{s,t} < \infty \rangle \]