Let \( D \) be a 1-dimensional Noetherian domain, then make \( \text{Spec}(D) =: X \) into a topological space by defining the closed sets \( \mathfrak{p} \) for prime \( \mathfrak{p} \geq \mathfrak{a} \) for any ideal \( \mathfrak{a} \) in \( D \).

For applications to arithmetic, too coarse. Consider pair \((X, O_X)\) where

\[ O_X = \text{the sheaf of rings given by} \]

\[ \mathcal{F} : U \rightarrow \mathcal{O}(U) = \{ f \mid g \equiv 0 \pmod{\mathfrak{p}} \forall \mathfrak{p} \in \mathfrak{a} \} \]

open, non-empty

"structure sheaf on \( \text{Spec}(D) = X \)" together with natural map

\[ \mathcal{O}(U) \rightarrow \mathcal{O}(V) \]

if \( V \subseteq U \) (if \( g \equiv 0 \pmod{\mathfrak{p}} \forall \mathfrak{p} \in \mathfrak{a} \))

induced by projection \( \mathfrak{p} \rightarrow \mathfrak{p} \cap \mathfrak{a} \).

**Terminology for sheaves:**

- elements in ring \( \mathcal{F}(U) \) are "sections" — def. of sheaf is that these sections are well behaved with respect to any open covering of open set \( U \).

"stalk" at a point \( x \in X \):

\[ F_x := \lim_{U \ni x} \mathcal{F}(U) \]

so elements of stalk are equivalence classes of sections

\[ S_U \sim S_V \text{ if we can find } W \supseteq U \cap V \text{ with } x \in W \]

s.t. \( S_U \mid_W = S_V \mid_W \) (i.e. apply restriction map to \( W \))

call these "germs" of sections at \( x \).

**Fact:** stalk of \( O_X \) at \( \mathfrak{p} \) is \( \mathcal{O}_{\mathfrak{p}} \).

(follows from definition. \( U = X \setminus \{ \mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n \} \forall \mathfrak{p}_i \) any \( j \) and \( \mathcal{O}_{\mathfrak{p}} = \{ f \mid g \equiv 0 \pmod{\mathfrak{p}} \} \) with natural inclusion \( \mathcal{O}(U) \rightarrow \mathcal{O}_{\mathfrak{p}} \).
Example 1: If $\mathcal{O}$ is DVR, then $\text{Spec}(\mathcal{O}) = \{ \mathfrak{m}, (0) \}$ is unique maximal ideal.

$\mathfrak{m}$ - closed pt., $(0)$ - generic point.

not closed, its closure is total space $X.$

So closed sets: $\emptyset, \{ \mathfrak{m} \}, X \Rightarrow$ open sets: $X, (0), \emptyset.$

and "functions" on $\mathcal{O}$ are elements $f$ with "values" $f \mod \mathfrak{m}, f \in \text{Frac}(\mathcal{O})$.

Example 2: If $\mathcal{O}$ is Dedekind domain, $\text{Spec}(\mathcal{O}) = \{ \mathfrak{p} \mid \mathfrak{p} \text{ prime} \}$.

Now $\mathcal{O}_{\mathfrak{p}}$ is a DVR with inclusion $\mathcal{O} \hookrightarrow \mathcal{O}_{\mathfrak{p}}$ with induced map

$f: X_{\mathfrak{p}} = \text{Spec}(\mathcal{O}_{\mathfrak{p}}) \rightarrow X = \text{Spec}(\mathcal{O})$.

Claim: $f$ is morphism of affine schemes. Affine scheme is pair $(X = \text{Spec}(A), O_X : \text{structure sheaf})$.

Any homom. of rings $\phi: \mathcal{O} \rightarrow \mathcal{O}'$ induces map on prime ideals

$f: X' \rightarrow X$, continuous, and corresponding map

$\mathfrak{p}' \mapsto \phi^{-1}(\mathfrak{p}')$.

$f_{\mathcal{U}}^*: O(\mathcal{U}) \rightarrow O(\mathcal{U}')$ when $\mathcal{U}' = f^{-1}(\mathcal{U})$.

\[ s \mapsto s \circ f|_{\mathcal{U}}. \]

with (1) for $V \subseteq \mathcal{U}$ open

\[ O(V) \xrightarrow{f|_V^*} O(V'). \]

(2) for $\mathfrak{p}' \in \mathcal{U}' \subseteq X'$, and $a \in O(\mathcal{U})$

\[ a(f(\mathfrak{p}')) = 0 \Rightarrow f_{\mathcal{U}}^*(a) \equiv 0 \mod \mathfrak{p}'. \]

i.e. $a \mod f(\mathfrak{p}') = 0$.

Not so easy to prove these properties!

Also can be shown all such morphisms are induced from homs. of rings.
with stalk at $\mathfrak{p}$ in $X$ equal to $\mathcal{O}_{\mathfrak{p}}$. "Germ of functions" in infinitesimal nbhd of $\mathfrak{p}$.

The set $\mathcal{M}_{\mathfrak{p}} = \{ f \in \mathcal{O}_{\mathfrak{p}} | f \neq 0 \}$ is not defined on nbhd. of $\mathfrak{p}$ in $X$, which will contain other primes if $X$ is not itself a local ring.

But any particular $f/g$ has nbhd. on which it is defined.

in $\mathcal{O}_{\mathfrak{p}}$ (require that $\mathfrak{o} \in U$ s.t. $g \neq 0 \mod \mathfrak{p}$)

Claim: for an order $\mathfrak{O}$, then if $\mathfrak{p}$ regular, so $\mathcal{O}_{\mathfrak{p}}$ DVR then curve non-singular.

But if $\mathfrak{O}_{\mathfrak{p}}$ not a DVR, where maximal ideal $\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$ not generated by single elt., then $\mathfrak{p}$ "singular"

Better to see from geometric setting, reason back to algebraic setting.

$C[x], C[x_1,y]/y^2 = x^2 + x$ are smooth, but $C[x_1,y]/y^2 = x^2 + x^2$ or $y^2 = x^2$ are singular.

Remember points on these varieties are max. ideals containing $I$: quotient ideal.

So $C[x] : (x-a) \mapsto a \in C$ $C[x_1,y]/E : (x-a, y-b) \mod E : y^2 = f(x) \mapsto (a,b) \in C^2 s.t. b^2 = f(a)$

Draw real locus: say of $b^2 = a^3$ or $b^2 = a^3 + a^2$. 

\[ \begin{array}{c}
\begin{array}{c}
\text{b} \\
\text{a}
\end{array} \\
\end{array} \]
To understand when these varieties are singular, Hartshorne would say analyze $M/M^2$ where $M$ is maximal ideal of $O_x$ in localization $O_x$

More precisely, we compute dimension of $M/M^2$ as $O_x/M$-vector space.

Then $O$ is "non-singular" at $x$ if $\dim_{O_x/M}(M/M^2) = \dim(O) = 1$.

(Atiyah-Macdonald tell us that, as a consequence of Nakayama's lemma, if $x_i$ are basis for $M/M^2$ as $O_x/M$-vector space if $M$ is $O_x$ module, $O$ local ring, then $x_i$ generate $M$. So suffices to analyze $M/M^2$ to find gens. for $M$.)

Taking $M=M$ in above statement.

Look at the point $(0,0)$ in our three examples:

For each, consider ideal $(x-0, y-0)$ in $\mathbb{C}[x,y]/E$.

$M^2 = \langle x^2, y^2, xy \rangle$ so $x \equiv y^2 - x^3 \pmod{E}; y^2 = x^3 + x \equiv 0 \pmod{M^2}$

so $M/M^2$ generated by $y$.

For other two examples, no relation mod $M^2$ on $x$ or $y$.

But say $(1,1)$ is non-singular on $y^2 = x^3$ since $y-1 = \frac{1}{2} (x-1)^3 + \frac{3}{2} (x-1)^2 - \frac{1}{2} (y-1)^2 + \frac{3}{2} (x-1) \pmod{E}$

$\equiv \frac{3}{2} (x-1) \pmod{M^2}$