Return to setting of general field, $K$, and again use dichotomy of valuations — archimedean v. non-arch. — to study them.

If $v$ with assoc. $1 \cdot v$ is non-archimedean then by 3 axioms for non-arch. valuation

know $\Theta = \{ x \in K \mid v(x) > 0 \} = \{ x \in K \mid |x|_v \leq 1 \}$

is subring of $K$ with units

$\Theta^* = \{ x \in K \mid |x|_v = 1 \}$ and unique maximal ideal $\mathfrak{m} = \{ x \in K \mid |x|_v < 1 \}$

Moreover, $\Theta$ is integral domain (since it is) with field of fractions $K$

where either $x \in \Theta$ or $x^{-1} \in \Theta$.

"valuation ring"

Fact: $\Theta$ is integrally closed. Thus if $K = \# field$, then $\mathbb{Z} \subseteq \Theta_v$ so

$(\text{in } \text{Frac}(\Theta) = K)$

$v: \text{valuation (non-arch.)}$

$\therefore \Theta_K \subseteq \Theta_v$

pf: Any elt $x \in K$ integral satisfies monic equation over $\Theta$

$x^n + a_1 x^{n-1} + \cdots + a_n = 0$ with $a_i \in \Theta$. Want to show $x \in \Theta$.

If not, then since $\Theta$ valuation ring, $x^{-1} \in \Theta$. But then

$x = -a_1 - a_2 x^{-1} - \cdots - a_n x^{-(n-1)} \in \Theta$. \(\triangleright\)

Examples:

$K = \mathbb{Q}$, $v \xrightarrow{} p$: prime, then $\Theta_{\mathbb{Q} v} = \mathbb{Z}(p) = \{ \frac{a}{b} \mid p \nmid b \}$

(similarly for $\#$ fields)

$K = \mathbb{Q}_p$, then $\Theta = \mathbb{Z}_p$. 

localization at $p$. 
Say that valuation is "discrete" if it admits smallest positive value \( m \).

Then the set of all possible valuations is \( m \mathbb{Z} \) for \( m \) of \( K \).
Always find equivalent valuation with \( m=1 \). ("normalized" valuations)

Note that sets \( 0,0^\times,\mathfrak{p} \) are independent of representative in equivalence class.

**Final Proposition:** if \( v \) is discrete then valuation ring \( \mathcal{O}_v \) is P.I.D.

( so \( \mathcal{O}_v \) is discrete valuation ring ) with \( g^n/g^{n+1} \cong \mathcal{O}_v/\mathfrak{p} \forall n \).

Moreover the chain of ideals \( 0=\mathfrak{p}^2 \mathfrak{p}^2 \cdots \) form a basis of open nbhds of 0 in \( K \). \( \{ g^n = \{ x \in K | |x|_v \leq \frac{1}{g^n-1} \} \) if

\[ 1 \mathfrak{p} \mathfrak{p} \] give base of nbhds

of 1 in \( 0^\times \).

**Archimedean valuations:** Given \( K \) field, any valuation \( v \), form completion \( \hat{K} \).

if \( v \) archimedean, not many choices for \( \hat{K} \).

**Theorem (Ostrowski)** \( K \) field, \( \hat{K} \) completion w.r.t. archimedean \( v \),

then there is an isomorphism \( \delta: \hat{K} \to \mathbb{R} \) or \( \mathbb{C} \)

such that \( |a|_v = |\delta(a)|_\infty \) \( \delta \) arch. on \( \mathbb{R} \) or on \( \mathbb{C} \) with \( \delta(0,1) \).
We may extend valuations to the completion just as for \( \Theta \), setting

\[
\hat{\nu} \text{ on } \hat{K} \text{ to be given by } \hat{\nu}(a) = \lim_{n \to \infty} \nu(a_n), \text{ if } a = \lim_{n \to \infty} a_n \text{ in } K, a \in \hat{K}
\]

"ultrametric" property \( \Rightarrow \hat{\nu}(a) = \nu(a_n) \) if \( n \geq n_0 \), some \( n_0 \).

Earlier we showed \( \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p / p\mathbb{Z}_p \) and \( \mathbb{Z} / p^n\mathbb{Z} \cong \mathbb{Z}_p / p^n\mathbb{Z}_p \), \( n \geq 1 \).

and same proof works for general valuation rings \( R \leq \Theta = \text{val. ring of } \hat{K} \) with \( \mathfrak{p}, \mathfrak{q} \) resp. maximal ideals.

**Proposition:** \( \nu \) discrete valuation on \( K \) with valuation ring \( \Theta \) \( R \leq \Theta = \text{set of reps for } \Theta / \mathfrak{p} : \text{residue field } (0 \in R) \)

Then \( x \in \hat{K} \setminus \Theta \) has unique power series rep in \( \Theta \):

\[
x = \pi^m \cdot (a_0 + a_1\pi + \ldots) \quad a_i \in \mathbb{R}, a_0 \neq 0, m \in \mathbb{Z}
\]

(convergent power series as all formal power series are Cauchy)
in non-arch. case.

**Example:**

1. \( K = \mathbb{Q} \), \( \mathfrak{p} = \text{max. ideal for } \nu_p = p\mathbb{Z} \), just get usual

\( p \)-adic expansion in \( \Theta_p \):

\[
x = p^m (a_0 + a_1p + \ldots)
\]

2. \( K = \mathbb{F}_q((t)) \) with \( \Theta = \mathbb{F}_q[t] \) \( \mathfrak{p} = (t-a) \) \( a \in \mathbb{F}_q \)

then \( \hat{K} : \text{completion w.r.t. } (t-a) \) is "field of formal power series" \( \mathbb{F}_q((\mathfrak{p})) \)

consisting of formal Laurent series \( f(t) = (t-a)^m (a_0 + a_1t + \ldots) \)

There is even analogous result saying

\[
\Theta = \lim_{n \to \infty} \Theta / \mathfrak{p}^n \quad (\text{postpone for next time})
\]
Given a valuation over \( \hat{K} \), want to explain how to extend it to algebraic extension \( E | K \). Key tool: Hensel's Lemma.

A polynomial \( f(x) = a_0 + \cdots + a_n x^n \), \( a_i \in \mathcal{O} \): valuation ring of \( K = \hat{K} \).

is called "primitive" if \( f \not\equiv 0 \pmod{\mathfrak{p}} \). In terms of valuation, we could say \( |f| = \max \{ |a_0|, |a_1|, \ldots, |a_n| \} \).

Hensel's Lemma: if \( f \) primitive with

\[
\overline{f} = \overline{g} \cdot \overline{h} \pmod{\mathfrak{p}} \quad \overline{g}, \overline{h} \text{ rel. prime polys.}
\]

then \( f = gh \) in \( \mathcal{O}[x] \) where \( g, h \) polys with \( \deg(g) = \deg(\overline{g}) \)

\[
\deg(h) = \deg(\overline{h})
\]

and \( g \equiv \overline{g}, h \equiv \overline{h} \pmod{\mathfrak{p}} \).

Usual version of Hensel's Lemma:

if \( f(a) \equiv 0 \pmod{p}, \quad f'(a) \not\equiv 0 \pmod{p}, \quad a \in \mathbb{Z}_p, \quad f \in \mathbb{Z}_p[x] \)

then \( \exists \, \alpha \in \mathbb{Z}_p \) with \( f(\alpha) = 0 \) and \( \alpha \equiv a \pmod{p} \).

(idea: lift the solution to higher and higher powers of \( p \), making formal sense.)

Example: \( x^2 - 7 \) in \( \mathbb{Z}_3 \). so \( 1 \) is sol'n in \( \mathbb{Z}/3\mathbb{Z} \)

How to lift it to sol'n mod \( 9 \)? \( 1 \) in \( \mathbb{Z}/9\mathbb{Z} \), not sol'n. Can lift to \( (1 + 3k) \)

\( k = 0, 1, 2 \).

e.g. \( (1 + 3) \) is lift to sol'n of \( x^2 - 7 \) in \( \mathbb{Z}/3^2\mathbb{Z} \)
as \( 16 - 7 = 0 \pmod{9} \).

General recipe for accomplishing lift is version of Newton's method.
Newbold's version is slight generalization since, if \( a \equiv \theta \) has
\[ f(a) \equiv 0 \mod \theta \] then we may write \( f(x) \equiv (x-a)h(x) \)
and condition that \( a \) is simple root (i.e. \( f'(a) \not\equiv 0 \mod \theta \))
guarantees that \((x-a)\) and \( h(x) \) are relatively prime \((\mod \theta)\).

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**proof of Hensel's lemma**: \( d = \deg(f) \), \( m = \deg(g) \), \( \deg(h) \leq d-m \)

If \( g_0, h_0 \in \Theta[x] \) are polynomials s.t. \( g_0 \equiv g, h_0 \equiv h \pmod{\theta} \)
then since \( g, h \) assumed relatively prime,

\( f(a(x), b(x)) \in \Theta[x] \) with \( a \cdot g_0 + b \cdot h_0 \equiv 1 \pmod{\theta} \)

Consider coeffs. of \( f - g \cdot h_0 \) and \( a \cdot g_0 + b \cdot h_0 - 1 \) \( \in \Theta[x] \)

Take one with smallest valuation, call it \( \pi \) (if min val = \( \infty \), we're done)

Try to find desired \( g, h \) among \( g = g_0 + p_1(x) \cdot \pi + p_2(x) \cdot \pi^2 \)

\[ h = h_0(x) + q_1(x) \cdot \pi + q_2(x) \cdot \pi^2 \]

with \( p_i \in \Theta[x] \), \( \deg(p_i) \leq m \)

\( q_i \in \Theta[x] \), \( \deg(q_i) \leq d-m \).

so that setting \( g_{n-1}(x) = g_0(x) + p_1(x) \cdot \pi + \ldots + p_{n-1}(x) \cdot \pi^{n-1} \)

then \( f \equiv g_{n-1} \cdot h_{n-1} \pmod{\pi^n} \)

Then, if this can be arranged, in limit as \( n \to \infty \), get \( f = g \cdot h \) \( \in \Theta[x] \)

(The ideal \( (\pi^n) \subset \Theta^n \), in particular, so \( (*) \) implies \( f \equiv g_{n-1} \cdot h_{n-1} \pmod{\pi^n} \))

Prove this for all \( n \) by induction.