Definition: finite ext'n \( L/K \) is "unramified" if residue fields of \( L \) and \( K \) are such that \( L/K \) is separable (automatically if finite) and

\[
[L : K] = [\lambda : K]
\]
i.e. \( e = 1 \) in identity \([L : K] = e \cdot f\)

Proposition: The composite of two unramified extensions is unramified.

Lemma: \( L' = (K' : L) \) then \( L/K \) unramified \( \Rightarrow \) \( L'/K' \) unramified.

Tower of extensions in fixed algebraic closure

pf of lemma: \( L/K \) is finite separable ext'n, so

\[\lambda = K(\overline{a})\]

with lift \( a \in O_L\).

min-poly for \( \lambda : f(x)/K \)

\( f(x) \) reduction in \( K[G] \)

claim that \( L = K(\alpha) \): indeed we have

\[\lambda = K(\alpha)\]

and \( \overline{f} \) is min-poly for \( \overline{\alpha}/K \).

in particular,\( L = K(\alpha), \quad [K(\alpha) : K] \leq [L : K] = [\lambda : K] \)

and \( \overline{f} \) is min-poly for \( \overline{\alpha}/K \)

Now \( L' = K'(\alpha) \)

If \( g \) min poly for \( \alpha/K' \)

\( g \) reduction in \( K'[\bar{x}] \)

then we have

\[\lambda : K' = \overline{f} \leq [L' : K'] = \deg g = \deg \overline{g} = [K'(\overline{a}) : K']\]

So if \( \overline{g} = \overline{h_1 \cdot h_2} \), know \( \gcd(h_1, h_2) = 1 \)

then by Hensel's Lemma

\( g \) is reducible \( \nRightarrow \) minimality of \( g \).

Conclusion: \( \overline{g} \) is irreducible, hence minimal for \( \overline{\alpha} \).

Key equality:

\( \overline{g} | \overline{f} \) as poly in \( K'[\bar{x}] \)

as \( \overline{g} \) is separable since \( \lambda/K \) separable ext'n.
This proves $L'/K'$ is unramified. In particular if $L = L'$ (special case) then $L'/K'$ unramified.

But then $K'/K$ must also be unramified by repeated use of degree identity $\deg(L'/K) = ef$ with $L: L'$ or $K: K'$ or $K$.

pf of proposition: If $L/K$ unramified, lemma implies $L/L'/L$. But then $LL'/K$ unramified since $L'/K$ unramified and can use degree is mult., separability is transitive.

(algebraic)

Definition: An infinite extension is unramified if all of it is union of unramified finite extensions.

Form composition of all unramified extensions in fixed alg. closure $\overline{K}$ of $K$.

"maximal unramified extension"

* In fact, proof of lemma can be used to show following classification of unramified extensions:

$L/K$ unramified $\iff$ $L = K(x)$ with $x \in \Theta$ s.t. $f = \text{red. of min. poly}$ in $K[t]$ is separable.

($\Rightarrow$) defin of unram. includes separability, so $f$ of residue field, lift to $x$.

Consider min. poly $g$, show $g = \overline{f}$

($\Leftarrow$) Hensel's Lemma: sep $\Rightarrow$ irreducible.
Proposition: If \(L/k\) finite extension, then \(L/k\) unramified (\(e = 1, \lambda/k\) separable)

\(<=\) \(L = K(x)\) for \(x \in O_L\) with \(\overline{f}_x \in kK_k[t]\) separable.

If \(L/k\) unramified \(\Rightarrow \lambda/K_k\) separable, so have primitive elt. thin.

Find \(x\) generating \(L/k\) over \(K_k\). Lift to \(\overline{x} \in O_L\).

Let \(g = \min \text{poly. of } x, \in O_K[t]\) since \(x\) integral over \(K\). (\(O_L\) is integral closure of \(O_K\) in \(L\)).

and \(\deg(g) = [L:K] = [\lambda_L:K_k]\) so \(\overline{g}\) must be minimal poly of \(\overline{x}\) (it satisfies \(\overline{g}(\overline{x}) = 0\) and \(\deg(\overline{g}) = \deg(g)\)).

In particular \(\overline{g}\) separable as \(\lambda_L/K_k\) separable.

\(<=\) If \(\overline{f}_x\) separable, then \(\overline{f}\) reducible \(\Rightarrow f\) reducible

\((\overline{f}_x: \text{reduction of min. poly. } f_x)\) with coeffs. \(\in O_K\)

so \(\overline{f}_x\) irreducible and so \(K_k(\overline{x})/K_k\) has degree \(= \deg(\overline{f}_x)\) is separable and \(= [L:K]\)

so \(L/k\) unramified. (in some fixed alg extn)

Result suggests how to create unram. extn \(L/k \leftrightarrow \text{sep. extn of residue field} K_k\).

Given \(K'/K_k = K_k(\overline{x})/K_k\) with min poly \(\overline{g}\). Lift: \(g\), monic

\(\overline{g}\) with root \(\overline{x}\) in \(K_k\) and \(x \equiv \overline{x} \mod e_{K_k}\)

Consider \(L = K[t]/g(t) = K(x)\) : unramified by proposition.
It is unique because, if $K', K''$ are unram. extensions of $K$
both with residue field $\lambda$, then the compositum $K' \cdot K''$ also
has residue field $\lambda$ and is unramified by earlier result.

Hence $[K' \cdot K'': K] = [\lambda : K \kappa] = [K' : K]$ so $K' = K''$.

Example: $K$: local field of char. 0 (finite extn of $\mathbb{Q}_p$)
then residue field has order $q = p^r$, some $r$. Finite extensions
of $K_\kappa$ are the splitting fields of $X^q - X$ over $K_\kappa$.

(Galois extns with cyclic Galois gp. generated by $x \rightarrow x^q$ "Frobenius elm.")

So by the above correspondence, unramified $K'$ extns of $K$
are likewise splitting fields of $X^q - X$ with cyclic Galois
$\mathbb{G}_p$ of order $r$ and canonical generator of "Frobenius"
with $\sigma(\alpha) = \alpha^q \pmod{\mathbb{F}_q}$ if $\alpha \in \mathcal{O}_K$

where $\mathfrak{f}_{\mathbb{F}_q}$: maximal ideal in $K_\kappa/K$

Can go further $\rightarrow$ "tamely ramified extns"

$L/K$ with $\lambda L/K_\kappa$ separable, $T$: max. unram. extn
of $K$. $\lambda L$ (by above $\leftrightarrow$)

$K_L$ in $\lambda L$.

Ask that $[L : T]$ is relatively prime to $p = \text{char}(K_\kappa)$
and similarly that compositum of tamely ram. is tamely ramified.