Thus every elt $x \in \mathcal{L} \cdot K_v$ is expressible as limit of elts in $\mathcal{L}$, i.e.

$$x = \lim_{n \to \infty} \mathcal{L} \cdot x_n, \quad x_n \in \mathcal{L} \cdot v_n$$

Since $\widetilde{\nu} \circ \mathcal{T} = \nu \circ \mathcal{T}'$ then \(\{\mathcal{T}' \cdot x_n\} \approx \{\mathcal{L} \cdot x_n\}\) converges in $\widetilde{\nu} \circ \mathcal{T}'$ topology of $\mathcal{L}' \cdot \mathcal{L} \cdot K_v$

Call resulting limit $\mathcal{L} \cdot x$.

Then $x \mapsto \mathcal{L} \cdot x$ is our desired isomorphism $\mathcal{L} \cdot K_v \to \mathcal{L}' \cdot \mathcal{L} \cdot K_v$.

and it leaves $K_v$ fixed. Now extend to autom. $\zeta$ of $\text{Gal}(\overline{K_v}/K_v)$.

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Fancy algebraic formulation: tensor products of vector spaces -

Have natural homomorphisms

$$\mathcal{L} \otimes_k K_v \to \mathcal{L} \otimes K_v$$

$q \otimes b \mapsto a \otimes b$ (or maybe better $\mathcal{T}(a) \otimes b$)

$L$ as $K$-vector space now viewed as $K_v$-vector space (extension of scalars)

If we do this for all $K_v$-embedding $v$ places $w$ over $v$

then have map

$$\phi: \mathcal{L} \otimes_k K_v \to \mathcal{T} \cdot \mathcal{L} \cdot v_w$$

Proposition: If $L/K$ separable, then $\phi$ is isomorphism.
proof of Proposition: Use earlier characterization with \( \alpha \) primitive elt, if its minimal polynomial.

\[
f(x) = \prod_{w \mid v} \text{image of } \alpha \text{ in } L_w \subseteq K_v[x] \]

view \( L \) as embedded in \( K_v \). (and \( L \cap w \))

Thus we have commutative diagram:

\[
\begin{array}{ccc}
K_v[x]/(f) & \xrightarrow{\sim} & \prod_{w \mid v} K_v[x]/(p_w) \\
\downarrow \alpha \times & & \downarrow \alpha \times \\
L \otimes K_v & \xrightarrow{\text{homom. above}} & \prod_{w \mid v} L_w \\
\end{array}
\]

for each \( w \mid v \)

since \( L = K[x]/(f) = K[\alpha] \)

and \( K_v \) extends scalars

Let \( \alpha w \): image of \( \alpha \in L \) in \( L_w \subseteq \overline{K}_v \).

we have \( L_w \cong L \cdot K_v \)

with \( p_w(x) \) min poly. of \( \alpha w \).

so \( [L_w : K_v] = \deg(p_w) \)

"local degree"

But since top and sides are isom.,

bottom must be isom. as well.

(explicitly sends \( \alpha \otimes 1 \mapsto \alpha w \)

for each \( w \mid v \))

Corollary:

\[
[L : K] = \sum_{w \mid v} [L_w : K_v]
\]

if \( L \mid K \) separable, finite

if \( v \) discrete, then since \( [L_w : K_v] = e_w f_w \), where \( e_w = [w(L^x) : V(K^x)] \)

\[
f_w = [\lambda_w : K_{K_v}]_L \]

Better notation:

\[
[L : K] = \sum_{w \mid v} e_w f_w
\]
If of corollary is immediate from proposition: 

\[ [L:K] = \dim_K (L) \]

\[ \text{dim}_K (L \otimes_K K) = \sum_{w|\nu} \dim_{K_{w}} (L_{w}) \]

\[ \text{prop.} \]

Compare to our ideal-theory fundamental identity:

\[ f \in \mathcal{O}_K : \text{Dedekind domain} \]

\[ f \mathcal{O}_L = \mathcal{O}_L^e_1 \cdots \mathcal{O}_L^e_r \]

\[ L|K : \text{finite extn} \]

and valuations \( w_i = \frac{1}{e_i} \nu_{\beta_i} \) are normalized extensions of valuation \( \nu = \nu_{\beta} \) on \( K \).

We said before that \( e_i \) indeed agree with ramification indices.

\[ [w_i(L^x) : v(K_x)] \]

Since in discrete valuation

\[ w_i(L^x) = w_i(\beta) \cdot \mathbb{Z}, \quad v(K_x) = v(\beta) \cdot \mathbb{Z} \]

What about \( f_i \)? Previously said

\[ f_i = \left[ \mathcal{O}_L/\mathcal{O}_L^{e_i} : \mathcal{O}_L/\mathcal{O}_L^{e_r} \right] \]

and when we pass to local fields these indeed are isomorphic to \( \mathcal{O}_L/\mathcal{O}_L^{e_i} \)

and \( \mathcal{O}_L/\mathcal{O}_L^{e_r} \).

Why does our factorization theorem before,

for all primes not dividing the conductor,

analyze \( f \cdot \mathcal{O}_L \)'s factorization by factoring min poly. \( f \) for \( \alpha \) s.t. \( L = K[\alpha] \).

mod \( p \):

\[ f(x) \equiv \mathcal{P}_1(x) \cdots \mathcal{P}_r(x) \pmod{p} \]

with \( \deg (\mathcal{P}_i) = \text{residual degree} \quad \text{of} \quad \alpha_i \), agree with

new factorization theorem?