Just as in the treatment via ideals, if $L/K$ Galois then can make more precise conclusions about extensions by studying action on valuations.

Always two cases - $L/K$ finite - familiar properties of Galois ext's

1-1 corresp. of intermediate fields and

subgrp. of $\text{Gal}(L/K)$

$L/K$ infinite - this 1-1 corresp. is false w/o modification.

Introduce Krull topology on $\text{Gal}(L/K)$, where base of open nbhds of $G$ consists of cosets $G \cdot \text{Gal}(L/M)$ with $M/K$ finite Galois subext. in $L/K$.

Then $\text{Gal}(L/K)$ is compact, Hausdorff in this topology and 1-1 corresp is restored by considering only closed subgroups w.r.t. Krull topology.

Proposition: $G$ acts transitively on the extensions $W$.

First note that $6 \in G = \text{Gal}(L/K)$ acts by $|\alpha|_W = 6^{-1}(\alpha)|_W$.

So $|\alpha|_W = 6^{-1}(\alpha)|_W = (6^*|_W)^{-1}$.

On: act on right: $|\alpha|_W 6 = (6^{-1}(\alpha)|_W = (6^*|_W)^{-1}$.

Either way, clearly $W_6$ extends $V$ since $6$ fixes $K$.

pf. of Proposition: In case of ideals, showed $\text{Gal}(L/K)$ acts transitively on $\mathfrak{g}$ in $G \mathfrak{O}_L = \mathfrak{g}_1 \cdots \mathfrak{g}_r$ by using fact that any two $\mathfrak{g}_i, \mathfrak{g}_j$ have $\mathfrak{g}_i \cap \mathfrak{O}_K = \mathfrak{g}_j \cap \mathfrak{O}_K = \mathfrak{g}$.

Now use Chinese Remainder Theorem to arrange $x \equiv 0 \mod \mathfrak{g}_i$ for all $\mathfrak{g}_i$ and $x \equiv 6^k \mod \mathfrak{g}_j$ for some $k$ from the given condition.

Thus, $x \equiv 6^k \mod \mathfrak{g}_j$ for all $\mathfrak{g}_j$, so $x$ extends $V$. 

Then taking norms, get \[ N_{Y/k}(x) \] both in \( p \), not in \( p \). 4.

\[
\text{If } 6(x) \quad \text{using } x=0 \quad \text{for} \quad \text{using } x=1 \mod 6(p_f) \\
\in \text{Gal}(Y_k)
\]

We do the same in case \( L/k \) finite, but recall that our replacement for CRT is:

Given inequivalent valuations on \( L \), \( e > 0 \), \( \infty \in L \)

we can find \( x \) s.t. \( |x-a_i| < e \forall i=1, \ldots , n \).

Indeed if \( w, w' \) not conjugate then \( \exists w \circ 6 \in \text{Gal}(Y_k) \) is completely disjoint from \( \exists w' \circ 6 \in \text{Gal}(Y_k) \), so by approximation there \( \exists x \in L \)

with \( |6x|_w < 1 \quad |6x|_{w'} > 1 \forall 6 \in \text{Gal}(Y_k) \).

Taking norms, let \( \alpha = N_{Y/k}(x) \). Then \( |\alpha v| = \prod 6 \times 6x \to 1 \)

\[
\text{If } 6 \in \text{Gal}(Y_k) \quad \text{and } 6 \in \text{Gal}(Y_{w'}) \quad |6x|_w > 1, \quad 6 \in \text{Gal}(Y_{w'})
\]

In infinite case, use above result + little topology:

if \( L/k \) infinite, \( w, w' \) vals.

Let \( M/k \) finite Galois subextfin. \( X_M = \{ 6 \in G \mid w_6 \circ M = w' \mid M \} \)

Know \( X_M \) non-empty by above, and in fact closed. \( X_M \) is trans.

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\( \text{Note: }\) if \( \bigcap X_M \neq \emptyset \)

since if \( 6 \in G \setminus X_M \) then open set \( 6 \circ \text{Gal}(Y/M) \) is in \( G \setminus X_M \), and so the complement of \( X_M \) is the union of open sets.

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Now if \( \bigcap X_M = \emptyset \) then since \( G \) compact, \( X_M \) closed \( \forall i.M \),

\( \Rightarrow \) \( \exists \) finite intersection \( \bigcap X_M = \emptyset \). \( \forall i = 1 \) since \( \bigcap X_M = X_{M_1 \cdot M_2 \cdots M_r} \)
the finite intersection claim is making use of Heine-Borel property.

Now we may define decomposition gp. assoc. to valuation \( w \) extending \( v \):

\[ G_w = \{ \sigma \in \text{Gal}(L'/K) \mid w \circ \sigma = w \} \]

If \( w, v \) non-archimedean (so have valuation, not just abs. value, with ring of

\[ I_w = \{ \sigma \in G_w \mid \sigma x \equiv x \mod \mathfrak{O}_L \forall x \in \mathfrak{O}_L \} \]

"inertia gp" - kernel of \( G_w \) under canon. homom. of \( L \) w.r.t. \( w \)

\[ R_w = \{ \sigma \in G_w \mid \frac{\sigma x}{x} \equiv 1 \mod \mathfrak{O}_L \forall x \in L^x \} \]

"ramification gp"

with \( w \circ \sigma = w \) implying that \( \sigma \) fixes \( \mathfrak{O}_L \) in particular, so \( I_w \)

is well-defined and that \( \sigma x \in \mathfrak{O}_L \forall x \in L^x \) so \( R_w \) well-defined.

Furthermore, \( G_w, I_w, R_w \) are closed in Krull topology and well-behaved

with respect to commutative diagrams:

\[ \begin{array}{ccc}
L & \xrightarrow{\tau} & L' \\
\downarrow \quad \tau & & \quad \downarrow \tau' \\
K & \rightarrow & K'
\end{array} \]

such that, for some homom. \( \tau \), the following

\[ \text{diagram commutes (on } L, \text{ extending one on } K) \]

If \( \tau \) isomorphism, then all induced maps are isoms.

Special case: \( \tau \) itself Galois autom. \( G_{w \circ \tau} = \tau^{-1}G_w \tau \) etc.
Can also apply diagram to case of tower with completions:

\[ \begin{array}{c}
L \\
\downarrow \quad \downarrow \\
L_w \\
\downarrow \\
K_v \\
\downarrow \\
K
\end{array} \]

then get map \( \text{Gal}(L_w/K_v) \to \text{Gal}(L/K) \)

\[ 6 \mapsto 6|_L \]

since homom. \( \tau : L \to L_w \) is inclusion.

So \( \tau^{-1} \) = restriction.

**Proposition:**

\[ \begin{align*}
G_w(L/K) & \cong G_w(L_w/K_v) \\
I_w(L/K) & \cong I_w(L_w/K_v) \\
R_w(L/K) & \cong R_w(L_w/K_v)
\end{align*} \]

**If:** Key fact is that \( G_w(L/K) \) is continuous with respect to \( W \).

6 continuous w.r.t. \( W \). Immediate that \( 6 \in G_w(L/K) \) is continuous w.r.t \( W \).

If \( 6 \in \text{Gal}(L/K) \), continuous, then

\[ |x|_W < 1 \Rightarrow \exists \{ x^n \} \to 0 \text{ in } W_{-\text{top}} \]

\[ \Rightarrow \exists 6x^n \to 0 \text{ in } W_{-\text{top}} \]

\[ \Rightarrow 6|x|_W < 1 \quad \text{i.e. } \quad |x|_{W_06} < 1 \]

\[ \Rightarrow W = W_{06} \quad \text{(and in fact equal since they agree on } K) \]

i.e. \( 6 \in G_w(L/K) \).

But now \( L \) dense in \( L_w \), so each \( 6 \in G_w(L/K) \) admits unique extension to continuous \( G_w(L_w/K_v) \) (also preserves \( I_{w(R_w)} \))