Infinite Galois theory / Formalism of class field theory in a way that applies equally well to local/global fields (see that better fit is with local fields, whose structure is much simpler.)

Last week, if our Galois ext' $L/k$ is infinite, gave the resulting Galois $G$ the Krull topology, with base of open sets of $G$ in $\text{Gal}(L/k)$ is

\[ \text{6. } \text{Gal}(L'M) \ni M: \text{finite, Galois over $k$ subject to $L$.} \]

Stated but didn't prove that $\text{Gal}(L/k) = G$ is compact, Hausdorff, with $1:1$ correspondence between subfields and closed subgroups of $G$.

Compactness: $G \to \text{Gal}(L'/k) \to \text{compact upon}$ giving discrete topology to all finite groups.

Note in Krull topology, taking $M$: Galois

(i.e. $M$ normal) so $\text{Gal}(L'/M)$ is a normal subgroup of $\text{Gal}(L'/k)$.

Motivated by this, define "profinite gp." to be a topological gp. Which is Hausdorff, compact, and is base of nbhds of 1 if its normal subgps.

Typical Neukirch definition: uses properties desired rather than explicit construction.

But as usual, can realize it by algebraic construction: projective limit. (see before in context of $\mathbb{Z}/p^n$, etc.)

Prop.: $G$ profinite, then $G \cong \lim_{\leftarrow N} G/N$, $N$: open normal subgps.

and conversely $\lim_{\leftarrow i} G_i =: G$ is profinite for any projective system $\{G_i\}$.
so in our example of profinite $\text{Gal}(L/k)$, then the proposition gives

$$\text{Gal}(L/k) = \lim_{\rightarrow} \text{Gal}(M/k) \quad \text{since} \quad \text{Gal}(M/k) = \frac{\text{Gal}(L/k)}{\text{Gal}(L/M)}$$

Simplest concrete example: $K = \mathbb{F}_p$. $\mathbb{F}_{p^n} / \mathbb{F}_p$ give projective system with Galois gps $\cong \mathbb{Z}/n \mathbb{Z}$

Hence $\text{Gal}(\overline{\mathbb{F}_p} / \mathbb{F}_p) = \lim_{\rightarrow} \text{Gal}(\mathbb{F}_{p^n} / \mathbb{F}_p)$

"absolute Galois gp" $= \lim_{\rightarrow} \frac{\mathbb{Z}}{n \mathbb{Z}} =: \hat{\mathbb{Z}}$

Neukirch does 9 examples in Section 2. Plan: Develop formalism of CFT through abstract profinite gps.

Always want to keep main example of Galois gp in mind. So index closed subgroups by set called "fields" $G_k : \text{closed } \subset G$, $k$ field. ("fixed field" of $G_k$) with $k$ s.t. $G_k : \text{"base field"}, \overline{k} : \text{split}. \overline{G_k} = \hat{\mathbb{F}_p^g}$, write $k \leq L$ for "fields" formally if $G_L \leq G_k$, with $L/k$ "finite" if $G_L$ of finite index in $G_k$, i.e. open and index is formal degree.

Study $\mathbf{G}$-modules of form: $\mathbf{A} : \text{mult. abelian gp}$ (e.g. mult. gp of field)

$\mathbf{G}$ acts like Galois autom. $: \xi \in G : a \mapsto a^\xi$

Since $G$ has topology, want action to be continuous: $G \times \mathbf{A} \to \mathbf{A}$ is continuous map when $\mathbf{A}$ is given discrete topology. $(\xi, a) \to a^\xi$

Find, for any $(\xi, a)$, an open subgp $U = G_k$ of $G$ such that open set $\{ U \times \xi a^\xi \}$ of $(\xi, a)$ is mapped to the open set $\xi a^\xi$, i.e. $a^\xi \in U$ etc. fixed by $U$. 
Since $A^G_k = A^{G_k}$ with $K/k$ finite then we can guarantee this if

we assume $A = U_{[K:k]<\infty} A^{G_k}$. Then any open set in $A$ consists of union of pts, each in some $A^{G_k}$ with inverse image open.

If $L/K$ extension of fields $A^{G_k} \subseteq A^{G_L}$. If, in particular, $L/K$ finite, then there is a norm map: $N_{L/K} : A^{G_L} \rightarrow A^{G_k}$ with

$$a \mapsto \prod a^6$$

where $a$ varies over reps of $G_L \backslash G_k$.

If $L/K$ Galois, then $A^{G_L}$ is a $\text{Gal}(L/K)$-module, with $(A^{G_L})^{\text{Gal}(L/K)} = A^{G_k}$.

Two key groups in formal class field theory:

1. $A^{G_k}/N_{L/K}(A^{G_L})$ "norm residue $G_k$"

   $$= H^0\left( \text{Gal}(L/K), A^{G_L} \right)$$

2. $A^{G_L}/I_{\text{Gal}(L/K)} A^{G_L}_{(1)} = H^{-1}(\text{Gal}(L/K), A^{G_L})$

   Where $A^{G_L}_{(1)} = \{ a \in A^{G_L} \mid N_{L/K}(a) = 1 \}$ "norm-one $G_L$"

   $$I_{\text{Gal}(L/K)} A^{G_L}_{(1)} = \langle a^6, a^{-1} \mid a \in A^{G_L} \rangle$$

Assume $G, A$ chosen such that the following is satisfied:

Axiom: $H^{-1}(\text{Gal}(L/K), A^{G_L}) = 1$ for all finite extensions $L/K$.

Then we establish several 1-1 correspondences, ingredients in later statements of class field theory.
Final ingredient is surjective $G$-homom. $f : A \to A$

with cyclic kernel $\ker f$.

Notation meant to suggest most important special case: $n$-th power map

where $\ker f = \ker n$, $n$-th rts. of unity.

In general, $\# \ker f$ is called the "exponent" of $f$.

Use this to define Kummer extensions w.r.t. $f$:

Fix $K$ s.t. $\ker f \leq A^G$. For every $B \leq A$, let

$K(B)$ be the fixed field of closed subgp. $H := \{ a \in G : a^b = b \quad \forall b \in B \}$

In particular $K(B)$ is Galois extn $/ K$ if $B$ is $G$-invariant.

Then Kummer extn is just special case $B = f^{-1}(\Delta)$ for some subset $\Delta \subseteq A^G$.

Def'ns abelian Galois extn of exponent $n_2$ as

$\text{Gal}(K(f^{-1}(\Delta))/K) \to \ker f$ is injective

$\alpha \mapsto \alpha^{n_2 - 1}$ where $\alpha \in f^{-1}(a)$

so $\text{Gal}(K(f^{-1}(\Delta))/K) \to \prod_{\Delta} \text{Gal}(K(f^{-1}(a))/K) \to \ker f^\Delta$

is injective homom into abelian gp.
Converse is also true: if \( L/K \) an abelian extension with exponent \( n \) (so \( L^n = 1 \) \( \forall \sigma \in \text{Gal}(L/K) \))
then \( L = K(\sigma^{-1}(\Delta)) \) with

\[
\Delta = A^g_L \cap A_k \\
\cong (A^g_L)^g \cong \overline{A}^g_k
\]

For some homomorphism \( \sigma \) with "exponent" \( n \).

If \( L/K \) cyclic then \( L = K(\langle \alpha \rangle) \) with \( \alpha^g = a \in A^g_k \).

Main Thm. of Kummer th.: The map \( \Delta \rightarrow L = K(\sigma^{-1}(\Delta)) \)
gives 1-1 correspondence between groups \( \Delta \) s.t. \( (A^g_k)^g \leq \Delta \leq A^g_k \)
and abelian ext's of exponent \( n \).

If \( \Delta \rightarrow L \) then \( A^g_L \cap A_k = \Delta \) and \( \exists \) canonical isomorphism.

\[
\Delta / A^g_L \cong \text{Hom}(\text{Gal}(L/K), \mathbb{Z}/g) \\
\text{a mod } A^g_k \mapsto [\chi_a : b \mapsto a^{b-1}] \\
\text{with } \chi_a \in \sigma^{-1}(a).
\]

Primary example: \( G := \text{Gal}(\overline{K}/K) \), \( A = \overline{K}^\times \) mult. gp. of alg. closure.

\( \sigma : a \mapsto a^n \) \( \gcd(n, \text{char}(K)) = 1 \)
(\( \text{arb. if char}(K) = 0 \))

Thus our axiom: \( L/K \) finite extension, then

\[H^1(L, \text{Gal}(L/K), (\overline{K}^\times)^{G_L}) = 1\]
is famous theorem:
"Hilbert 90".
Corollary: \( n \in \mathbb{N} \), \( \gcd(n, \text{char}(K)) = 1 \). Suppose \( \mu_n \in K \).

Then abelian extensions of exponent \( n \) \( \xrightarrow{\text{1-1}} \) \( \Delta \subseteq K^\times \) with \((K^\times)^n \subseteq \Delta \)

\( \frac{L}{K} \)

via the map \( \Delta \mapsto L = K(\sqrt[n]{\Delta}) \) and \( \text{Gal}(\frac{L}{K}) \cong \text{Hom}(\Delta/(K^\times)^n, \mu_n) \)

adjoin \( n \)th roots of 

\( \text{elts} \) of \( \Delta \).