Main Theorem of Class Field Theory: For $L/K$ finite, Galois

$\eta_L : \text{Gal}(L/K)^{ab} \to A_K / N_{L/K} A_L$

where $G^{ab} = G/[G,G]$

is an isomorphism, provided that

$| H^i(\text{Gal}(L/K), A_L) | = \mathfrak{S} \left\{ |L_i : K| \right\}$ for $i = 0,$

$1$ for $i = -1$. Some quotient of elements in $A_L$ of norm 1.

Corollary: $\eta_L : N_{L/K} A_L$ gives 1-1 correspondence between finite abelian extentions $L$ and open subgroups of $A_K$. (Topology: a has base of nbhds $a \cdot N_{L/K} A_L$, L: finite Galois)

With good fine-tuned properties. E.g.

$L_1 \leq L_2 \iff N_{L_1/K} A_{L_2} \supseteq N_{L_2/K} A_{L_1}.$

Local class field theory: $A_K = K^x$ so $A_K / N_{L/K} A_L = K^x / N_{L/K} L^x$

Given $\sigma \in \text{Gal}(L/K) \mapsto \tilde{\sigma} \in \text{Gal}(\tilde{L}/\tilde{K})$ i.e. $\tilde{\sigma} |_{\tilde{K}} = \sigma^n$, some $n 

s.t. \quad d_K(\tilde{\sigma}) \in \mathcal{N}$

$\mapsto N_{\tilde{L}/\tilde{K}}(\pi_\tilde{L})$ where $\mathcal{N} = \text{fixed field of } \tilde{\sigma}$.

LCFT Corollary 1: Every finite abelian extension $L/Q_p$ is contained in $Q_p(\xi)$, $\xi$: rt. of unity

LCFT Corollary 2: Every finite abelian extension $L/Q$ is contained in $Q(\xi)$, $\xi$: rt. of unity

(Kronecker-Weber)

Then

Can check that "norm topology" in abstract CFT $\leftrightarrow$ usual valuation theory topology for local fields.
proof of Corollary 1: Find \( f, n \) such that \((p^f) \times U_{\mathfrak{D}p}^{(n)} \leq N_{\mathfrak{D}p} L^x \) \((\ast)\)

By functoriality of CFT, the "class field" \( M \leftrightarrow \text{open subgroup} (p^f) \times U_{\mathfrak{D}p}^{(n)} \) contains \( L \)

But \((p^f) \times U_{\mathfrak{D}p}^{(n)} = (p^f) \times U_{\mathfrak{D}p} \cap (p) \times U_{\mathfrak{D}p}^{(n)} \)

so \( M \) is the composition of fields corresponding to each of the subgroups.

A \((p^f) \times U_{\mathfrak{D}p}^{(n)} \leftrightarrow \mathfrak{D}_p (\mathbb{Z} / m^n) / \mathfrak{D}_p \)

B \((p^f) \times U_{\mathfrak{D}p} \leftrightarrow \text{unramified extn of deg} f. \quad M \leq \mathfrak{D}_p \left( \frac{\mathbb{Z}_p}{(p-1)p^n} \right) \)

\((\ast)\) Fact that every open subgroup \( N \) of finite index in \( K^x \) contains such a subgroup, of form \( (\pi^f) \times U_{\mathfrak{D}k}^{(o)} \), \( n > 0, f > 0 \).

open subgp. of identity

\( 1 + p^n \)

and set \( U_{\mathfrak{D}k}^{(o)} = U_{\mathfrak{D}k} \).

Prop: \( L/K \) finit, abelian exthn is unramified \( \iff \quad U_{\mathfrak{D}k} \leq N_{\mathfrak{D}k} L^x \)

\((\Rightarrow)\) if \( M/K \) unramified, then degree \( n \)

\( N_{M/K} M^x = (\pi_{k^x}^n) \times U_{\mathfrak{D}k} \leq N_{\mathfrak{D}k} L^x \)

by assumption.

so \( L \leq M \), i.e. unramified.

\((\Leftarrow)\) \( L/K \) unram. \( \Rightarrow \) \( U_{\mathfrak{D}k} = N_{\mathfrak{D}k} U_{\mathfrak{D}k} \).

Thus in case B above, know corresponding field is unramified.

But result A is specific to \( \mathfrak{D}_p \).
proof of Corollary 2 from Corollary 1: $S := \text{finite set of primes ramified in } L$. Let $L_p$ be a completion of $L$ w.r.t. a prime above $p \in S$. As $L_p \mid L \cdot \mathcal{O}_p$, then $L_p / \mathcal{O}_p$ is abelian extn, so by

Corollary 1: $L_p \leq \mathcal{O}_p(\mu_{n_p})$ for some $n_p \in \mathcal{N}$. Let $p^{\mathfrak{e}_p} \parallel n_p$

and set $n = \prod_{p \in S} p^{\mathfrak{e}_p}$. Claim: $L \leq \mathcal{O}(\mu_n)$.

Proof of claim: Set $M = L(\mu_n)$. Then $M / \mathcal{O}$ abelian with $p$ ramified in $M / \mathcal{O}$ only if $p \in S$. $M_p$ completion of $M$ restricting to $L_p$.

So $M_p = L_p(\mu_n) = \mathcal{O}_p(\mu_{p^{\mathfrak{e}_p}}) \cdot \mathcal{O}_p(\mu_{n_p})$ where $n = p^{\mathfrak{e}_p} n'$ with $\gcd(n', p) = 1$.

Intra subgp. $I_p$ of $M_p / \mathcal{O}_p$ is $\text{Gal}(\mathcal{O}_p(\mu_{p^{\mathfrak{e}_p}}) / \mathcal{O}_p)$

since $\mathcal{O}_p(\mu_{n_p})$ is maximal unram extension of $\mathcal{O}_p(\mu_n)$.

so $I_p$ has order $\varphi(p^{\mathfrak{e}_p})$. Let $I = \langle I_p \rangle_{p \in S} \leq \text{Gal}(M / \mathcal{O})$

so that fixed field of $I$ is unramified extn of $\mathcal{O}$.

By a theorem of Minkowski, only unramified extn of $\mathcal{O}$ is $\mathcal{O}$ itself.

so $I \cong \text{Gal}(M / \mathcal{O})$. Thus $\# I = \# \text{Gal}(M / \mathcal{O})$

$\geq [\mathcal{O}(\mu_n) : \mathcal{O}]$.

On other hand:

$\# I \leq \prod_{p \in S} \varphi(p^{\mathfrak{e}_p}) = \varphi(n) = \prod_{p \in S} \varphi(n) = [\mathcal{O}(\mu_n) : \mathcal{O}]$. 

In general, we'd like to know class fields for important set of open subgroups \((\mathfrak{m}^f) \times \mathbb{Z}_K^{c_n}\). Theory of Lubin-Tate extensions.

These are extensions \(L_n = K(F(n))\) where \(F(n)\) evaluation of

(for \(f = 1\))

Take composition with unram. extn corresponding to \((\mathfrak{m}^f) \times \mathbb{Z}_K\)

to get open subgp. \((\mathfrak{m}^f) \times \mathbb{Z}_K^{c_n}\).

Examples: Ask for primes \(p\) s.t. \(K_f/\mathcal{O}_f\) has minimal poly \(f\)

where \(f\) splits completely into linear factors

Theorem: \(K_f > K_g\) iff \(\text{Spl}(f)\): primes split in \(K_f\)

where \(\subseteq^*\) means "up to finitely many exceptions"

\((\Rightarrow)\) easy: Multiplicativity of \(e_i, f_i\) in towers.

\((\Leftarrow)\) Chebotarev density theorem

Example: \(K_g = \mathcal{O}(\sqrt{p})\)

\(K_f^f = \mathcal{O}(\sqrt{p}), f \equiv 1 (4)\)

\(p\) splits completely

\(\Rightarrow \qquad (\frac{g}{f}) = 1\)

\(g\) splits completely

\(\Rightarrow \qquad g \equiv 1 (p)\)

Then residue field contains \(p^n\) roots of unity

\(K_f \supset K_g\)

since \(\text{Spl}(f) \supset \text{Spl}(g)\)

so indeed

\(\sqrt{p}\) is a

\(\mathcal{O}(\sqrt{p})\) by

Gauss sums.