Last time, proved Kronecker-Weber Theorem using main thm. of local CFT.

Guaranteed any abelian extn $L/k$ is contained in cyclotomic extn $\mathbb{Q}(\xi_n)$ (even understood that $n$ in terms of ramified parts of local extns.)

Do example showing how K-W thm can be used to determine splitting:

$L = \mathbb{Q}(\sqrt[p]{q})$. $p \equiv 1 \pmod{4}$. $\text{Disc}(L) = p$. (not $4p$).

What is $n$ s.t. $L \subseteq \mathbb{Q}(\xi_n)$? Could study ramification, or compute

$$\text{disc}(\mathbb{Q}(\xi_p)) = (-1)^{\frac{p-1}{2}} \sqrt[p]{p^{p-2}}$$

which is square of elt. in $\mathbb{Q}_p$.

so $\mathbb{Q}(\sqrt[p]{(-1)^{\frac{p-1}{2}} \sqrt[p]{p^{p-2}}}) \subseteq \mathbb{Q}(\xi_p)$

$s^2 \equiv p \pmod{p}$ if $p \equiv 1, 3 \pmod{4}$.

So $L \subseteq \mathbb{Q}(\xi_p)$, can't have any smaller $n$ since $p$ prime.

so have natural map: $$(\frac{\cdot}{p}) : \text{Gal}(\mathbb{Q}(\xi_p) / \mathbb{Q}) \to \text{Gal}(\mathbb{Q}(4\mathbb{Z}))$$

If $L = \mathbb{Q}(\sqrt[p]{q})$, $\text{Gal}(\mathbb{Q}(4\mathbb{Z}) / \mathbb{Q}) \cong \mathbb{Z}/4 \mathbb{Z}$

then have surjective homomorphism:

$$(\mathbb{Z}/4 \mathbb{Z})^* \longrightarrow \mathbb{Z}/4 \mathbb{Z}$$

with kernel the unique index 2 subgroup in our cyclic $q$th, the square classes.

Call it $I_L$.

so $a \longmapsto (\frac{a}{p})$ restricted to $L$.

When does a prime $q$ split completely? If $e_1 f = 1$ in factorization above $q$. 

If \( \mathfrak{b} \not\div \text{disc}(L) \), then \( e_{\mathfrak{b}} = 1 \) and \( f_{\mathfrak{b}} = \text{order of } \mathfrak{b}. I_L \) in \( (\mathbb{Q}/\mathbb{Z})^x \).

For us \( (\mathbb{Q}/\mathbb{Z})^x / I_L \cong \{ \pm 1 \} \) under hom. and sends \( \mathfrak{b} \mapsto (\frac{\mathfrak{b}}{p}) \).

i.e. \( \mathfrak{b} \) splits completely \( \iff \mathfrak{b} \neq p \) and \( \mathfrak{b} \) among quad res. mod \( p \).

so only depends on modulus \( p \), the order of \( \mathbb{F}_p \).

Nakrich proves Q.R. earlier in book using similar connection to cyclotomic fields.

Nice theorem on how set of primes that split characterize Galois extn:

\[
\text{Then: } K_f, K_g \text{ be normal extn corrsp. to min polys f.g.}
\]
\[
\text{Spl}(f), \text{Spl}(g) \text{ set of primes that split completely.}
\]

\[
\text{Then } K_f \not\subset K_g \iff \text{Spl}(f) \preceq^* \text{Spl}(g) \text{ where } \preceq^* \text{ means containment is up to finitely many exceptions.}
\]

\( H_f : (\Rightarrow) \text{ mult. of e if in towers.} \)
\( (\Leftarrow) \text{ Chebotarev density thm. - result of analytic techniques you'll learn next semester.} \)

In our example: \( \text{Spl} \left( L = \mathbb{Q}(\sqrt[p]{p}) \right) : (\frac{2}{p}) = (\frac{p}{2}) = 1 \)

\( \text{Spl} \left( \mathbb{Q}(\xi_p) \right) : \mathfrak{b} \equiv 1 \pmod{p} \) \( \mathfrak{b} \) has \( p \)th roots of unity.
To connect to Global CFT: Bundle together local fields:

\[ K : \text{Global field, } \mathcal{K} : \text{completions w.r.t. valuation } v . \]

\[ \mathcal{K} := \prod_v \mathbb{K}_v \quad \text{where ' means that elts in } \mathcal{K} \text{ must be a compact gp. } \mathcal{O}_v : \text{val. ring, } \]

\[ \text{for almost all } v . \]

\[ (\text{permits harmonic analysis with measure s.t. } \mu(\mathcal{O}_v) = 1) \]

\[ \mathcal{A}_k : \text{units of } \mathcal{A}_k \text{ - "idèles" } \mathcal{I}_k \]

\[ := \prod_v \mathbb{K}_v \quad \text{so ' means elts are units in } \mathcal{O}_v \text{ at almost all } v . \]

\[ \mathcal{K}^\times \text{ embeds diagonally } a \mapsto (a, a, \ldots) \in \mathcal{I}_k \text{ "principal idèles" } \]

and \[ \mathcal{I}_k / \mathcal{K}^\times : \text{idèle class gp. } (\text{ideal class gp, with extra info at infinite places}) \]

\[ \mathcal{I}_k / \mathcal{K}^\times \]\

this is module \( \mathcal{A}_k \) in our abstract class field theory framework.

do\( v \) maps come from products of local \( d_v \) maps.

(well-defined since elts live in restricted tensor products.)

\[ \text{Given ideal } \mathcal{M} := \prod \mathfrak{p}^{e_i} \sim \mathcal{M}^*_{\mathcal{K}} = \prod \mathcal{U}_{\mathfrak{p}}^{(e)} \text{ with } \mathcal{U}_{\mathfrak{p}}^{(e)} = \mathcal{U}_{\mathfrak{p}}. \]

Then \[ \mathcal{A}_k^* \sim \mathcal{I}_k / \mathcal{K}^\times \] are closed subgps of finite index in \( \mathcal{A}_k : \text{idèle class gp.} \)

If \( \mathcal{K} = \mathbb{Q} \), \( \mathbb{A}_\mathbb{Q} / \mathbb{A}_{\mathbb{Q}} = (\mathbb{Z}/\mathbb{m})^\times \cong \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}). \)
In general, there will be a class field \( K_m \leftrightarrow \mathfrak{A}_m \subseteq \mathfrak{A}_K \) in 1-1 correspondence of Global CFT sending abelian extins of \( K \) to closed subgps of finite index in \( \mathfrak{A}_K \).

so ray class fields of \( \mathfrak{A}_m \leftrightarrow \) cyclotomic fields
(closed subgps of finite index in \( \mathfrak{A}_K \))

Difficult open problem to find generating set of elts for ray class fields over #field \( K \). If \( K = \text{imag. quadratic} \), adjoin to \( K \) special values of the elliptic functions on the lattice of integers \( \mathcal{O}_K \subseteq \mathbb{C} \). 

"ray class field"