Non-Book Problem #1) We seek solutions to the equation $z^5 = 3 + 3i = 3\sqrt{2}e^{i\pi/4}$. Clearly, the absolute value of any such $z$ must be

$$|z| = \sqrt[4]{27*2} = \sqrt[4]{54}.$$ 

Furthermore, the argument of such a $z$ must satisfy

$$\arg(z) * 5 = \frac{\pi}{4} \pm 2\pi k \quad \text{where} \quad k \in \mathbb{Z}.$$ 

Thus, the possible values of $\arg(z)$ within $[0, 2\pi)$ are $\pi/20, 5\pi/20, 9\pi/20, 13\pi/20, \text{and} 17\pi/20$. Thus, the fifth roots of $3 + 3i$ are

$$\sqrt[5]{54}e^{i\pi/20}, \sqrt[5]{54}e^{5\pi i/20}, \sqrt[5]{54}e^{9\pi i/20}, \sqrt[5]{54}e^{13\pi i/20}, \text{and} \sqrt[5]{54}e^{17\pi i/20}.$$ 

1.1.4

3) First, we do some basic manipulation, using the assumption that we do not have $|a| = |b| = 1$, meaning $1 - \overline{ab} \neq 0$:

$$\left| \frac{a - b}{1 - \overline{ab}} \right| = 1$$

$$\Leftrightarrow \left| \frac{a - b}{1 - \overline{ab}} \right|^2 = 1$$

$$\Leftrightarrow (a - b)(a - \overline{b}) = (1 - \overline{ab})(1 - ab)$$

$$\Leftrightarrow |a| + |b| - \overline{ab} - b\overline{a} = 1 + |a||b| - \overline{a}b - \overline{b}a$$

$$\Leftrightarrow |a| + |b| = 1 + |a||b|.$$ 

Thus, if either $|a|$ or $|b|$ is 1, then the above expression is trivially true. Furthermore, if $|a| = |b| = 1$, then as long as $a \neq b$, we will have $1 - \overline{ab} \neq 0$, meaning the above manipulation is still valid. We can conclude that in all cases except $a = b$ and $|a| = 1$, if $|a| = 1$ or $|b| = 1$ we may say

$$\left| \frac{a - b}{1 - \overline{ab}} \right| = 1.$$ 

4) For this problem, let $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, and $z = z_1 + iz_2$. The equation

$$az + \overline{b}z = -c$$

gives us a system of two equations when we examine the real and imaginary parts separately:

$$(a_1 + b_1)z_1 + (b_2 - a_2)z_2 = -c_1$$
$$(a_2 + b_2)z_1 + (a_1 - b_1)z_2 = -c_2.$$
Thus, we will have a unique complex solution \( z = z_1 + iz_2 \) if and only if the above system is consistent and independent, that is, if and only if
\[
\det \begin{pmatrix} a_1 + b_1 & b_2 - a_2 \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix} \neq 0.
\]
As the determinant of the above matrix is
\[
a_1^2 + a_2^2 - b_1^2 - b_2^2 = |a|^2 - |b|^2,
\]
we conclude that the equation \( az + b\bar{z} + c = 0 \) will have exactly one solution whenever \(|a| \neq |b|\).

To solve for \( z_1 \) and \( z_2 \), we simply apply Cramer's rule:
\[
z_1 = \frac{-c_1(a_1 - b_1) + c_2(b_2 - a_2)}{|a|^2 - |b|^2},
\]
\[
z_2 = \frac{-c_2(a_1 + b_1) + c_1(a_2 + b_2)}{|a|^2 - |b|^2}.
\]

1.1.5

1) Using the same algebra as in 1.1.4 # 3 (but replacing \( = \) with \( < \)), we can see that
\[
\left| \frac{a - b}{1 - \bar{a}b} \right| < 1 \Leftrightarrow |a|^2 + |b|^2 < 1 + |a|^2|b|^2.
\]
Note that we can ignore the case where \( \bar{a}b = 1 \), as by assumption \(|a| < 1 \) and \(|b| < 1 \). We can rewrite the above expression as
\[
|a|^2 + |b|^2 - |a|^2|b|^2 < 1 \Leftrightarrow |a|^2(1 - |b|^2) + |b|^2 < 1.
\]
However, since \(|b| < 1 \), \( 1 - |b|^2 > 0 \), and since \(|a| < 1 \), we may say \(|a|^2(1 - |b|^2) < (1 - |b|^2)\).
Thus, in our situation,
\[
|a|^2(1 - |b|^2) + |b|^2 < (1 - |b|^2) + |b|^2 = 1,
\]
as desired.

4) For this problem, we freely use the fact that \(|z - a| = d(z, a)\) in the complex plane. By the triangle inequality, we know that
\[
d(z, a) + d(z, -a) \geq d(-a, a) \Rightarrow |z - a| + |z + a| \geq 2|a|.
\]
By the above expression, it follows that, for this equation to have a solution, it must be that \(|a| \leq |c|\), as otherwise the expression would violate the triangle inequality. By similar geometric logic, if \(|a| \leq |c|\), then there is at least one point (usually 2) in \( \mathbb{C} \) with distance \(|c|\) from both \( a \) and \(-a\). Any such point can be a solution to this equation.

Observe that if \(|a| = |c|\), then the only solution to this equation is to have \( z = 0 \), so it is possible for \(|z| = 0 \). On the other hand, the triangle inequality also tells us that \(|z| \leq |c|\).

Since setting \( a = 0 \) allows \( z = c \) as a solution, we conclude that \( 0 \leq |z| \leq |c| \).
1.2.2

5) As long as \( n \not\equiv h \),
\[
1 - \omega^h + \omega^2h - \cdots + (-1)^{n-1}\omega^{(n-1)h} = \sum_{k=0}^{n-1}(-\omega^h)^k
\]
is a finite geometric series with more than one summand. Thus,
\[
\sum_{k=0}^{n-1}(-\omega^h)^k = \frac{1 - (-\omega^h)^n}{1 + \omega^h} = \frac{1 - (-1)^n}{1 + \omega^h} = \begin{cases} 
0 & \text{if } n \in 2\mathbb{Z} \\
\frac{2}{1 + \omega^h} & \text{if } n \in 2\mathbb{Z} + 1
\end{cases}
\]

1.2.3

1) By our work in problem 1.1.4 #4, we know that it is sufficient to study the following system of equations (derived by studying the real and imaginary parts separately):
\[
(a_1 + b_1)z_1 + (b_2 - a_2)z_2 = -c_1
\]
\[
(a_2 + b_2)z_1 + (a_1 - b_1)z_2 = -c_2.
\]
This will have exactly one solution when the system is independent and consistent (which occurs when \(|a| \neq |b|\), and it will have zero solutions when the system is inconsistent. Thus, we need to study the situation where \(|a| = |b|\), and where not all of \(a, b,\) and \(c\) are zero (since then the entire plane is a solution). If we insist on those conditions, then the system will have a line as a solution when there exists a constant \(k \in \mathbb{R}\) such that
\[
c_1 = kc_2
\]
\[
a_1 + b_1 = k(a_2 + b_2)
\]
\[
b_2 - a_2 = k(a_1 - b_1).
\]

2) For this problem, we use that absolute value is a distance function, and appeal to the geometric definitions of ellipse, hyperbola, and parabola. Let \(f_1\) and \(f_2\) be two distinct points in \(\mathbb{C}\), and let \(c \in \mathbb{C}\) such that \(|c| > |f_1 - f_2|\). For any such \(c\), An ellipse with foci at \(f_1\) and \(f_2\) will be given by the points \(z\) satisfying
\[
|z - f_1| + |z - f_2| = |c|.
\]
Similarly, for any \(c \in \mathbb{C}\), the following equation will give a hyperbola:
\[
|z - f_1| - |z - f_2| = \pm|c|.
\]
Note that this allows some degenerate cases.
Finally, for a parabola, let \(a, b, f \in C\) where \(a \neq 0\). The following equation will give a parabola:
\[
|f - z| = \min_{t \in \mathbb{R}} |z - (a + bt)|.
\]
Here, \(f\) is the focus and \(a + bt\) is the directrix of the parabola.
1) \( \Rightarrow \) Assume \( \overline{zz'} = -1 \). Note that this implies \( \overline{zz'} = -1 \). \( z \) and \( z' \) will correspond to antipolar points on the sphere if the distance between those points (in \( \mathbb{R}^3 \)) is maximized, that is, when the distance is 2. Using the distance equation from class, we get that

\[
d(z, z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}
\]

\[\iff d(z, z')^2 = 4(z - z')(z - z') \]

\[= \frac{4|z|^2 + |z'|^2 + 1 + |z|^2|z'|^2}{|z|^2 + |z'|^2 + 1 + (zz')(zz')}
\]

\[= 4 = 4 \Rightarrow d(z, z') = 2 \]

Thus, the assumption that \( \overline{zz'} = -1 \) implies that the chordal distance between the corresponding points on the Riemann sphere is 2, so these points must be antipodal.

\( \Leftarrow \) Assume \( z \) and \( z' \) correspond to antipolar points on the sphere. Since \( \mathbb{C} \) is a field, there is a unique number \( x \in \mathbb{C} \) satisfying \( zx = -1 \). By part (a), \( x \) and \( z \) correspond to antipolar points. However, since each point on the sphere has exactly one antipolar point, it must be that \( z = z' \).

2) Observe that our cube must have vertices at

\[\{ c \ast (x_1, x_2, x_3) : x_1, x_2, x_3 \in \{-1, 1\}\}\]

for some nonnegative, real constant \( c \). By elementary geometry, we may see that \( c = 1/\sqrt{3} \) suffices. By the formula for a stereographic projection, we find the corresponding points in \( \mathbb{C} \) are

\[\left\{ \frac{x + iy}{1 - z} : x, y, z \in \left\{ -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\} \right\}.
\]