By definition, we have
\[ \cos i = \frac{e^{i\frac{\pi}{2}} + e^{-i\frac{\pi}{2}}}{2} = \frac{e^{-\frac{\pi}{2}} + e^{\frac{\pi}{2}}}{2} = \frac{1}{2e} + \frac{e}{2}. \]

Similarly, we compute
\[ \sin i = \frac{e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}}{2i} = \frac{1}{2i} - \frac{e}{2i} = -\frac{i}{2} + \frac{ie}{2} = i \left( \frac{e}{2} - \frac{1}{2e} \right). \]

Finally, we compute
\[ \tan(1 + i) = \frac{e^{i(1+i)} - e^{-(1+i)}}{e^{i(1+i)} + e^{-(1+i)}} = -i \frac{e^i e^{-1} - e^{-i} e}{e^i e^{-1} + e^{-i} e} = -i \frac{e^i - e^{-i}}{e + e^{-1}} = -i \frac{e^2 - 1}{e^2 + 1} = -i \frac{e^{2i} - e^2}{e^{2i} + e^2}. \]

---

2.3.4 #5 Find the real and imaginary parts of \( \exp(e^z) \).

If \( z = x + iy \), then we have
\[ e^z = e^{x+iy} = e^x e^{iy} = e^x \cos(y) + ie^x \sin(y) = e^x \cos(y) \cos(e^z \sin y) + i e^x \cos(y) \sin(e^z \sin y). \]

Therefore the real part of \( e^z \) is
\[ e^x \cos(y) \cos(e^z \sin y) \]
and the imaginary part is
\[ e^x \cos(y) \sin(e^z \sin y). \]
2.3.4 #6 Determine all values of $2^i$, $i^i$ and $(-1)^{2i}$.

Here goes more computation, where each $k$ ranges over the integers:

$$2^i = e^{i \log 2} = e^{i(\log |2| + i\arg 2)} = e^{i \log |2|} e^{-2\pi k}, \checkmark$$

$$i^i = e^{i \log i} = e^{i(\log |i| + i\arg i)} = e^{i \log |i|} e^{-\left(\frac{\pi}{2} + 2\pi k\right)} = e^{-\left(\frac{\pi}{2} + 2\pi k\right)}, \checkmark$$

$$(-1)^{2i} = e^{2i(\log |-1| + i\arg |-1|)} = e^{2i \log |1|} e^{-2(\pi + 2\pi k)} = e^{-2\pi + 4\pi k}, \checkmark$$

\[
2.3.4 \ #10 \ Show \ that \ the \ roots \ of \ the \ binomial \ equation \ z^n = a \ are \ the \ vertices \ of \ a \ regular \ polygon.
\]

The roots of the binomial equation are given by

$$z = \sqrt[n]{|a|} e^{i \frac{\arg a + 2\pi k}{n}},$$

where $1 \leq k \leq n$. Therefore, they all lie on a circle centered at the origin with radius $\sqrt[n]{|a|}$, and are equally spaced angles since the arguments differ by $\frac{2\pi k}{n}$. That is, they form a regular $n$-gon. \[\blacksquare\]
3.1.2 #1 If \( S \) is a metric space with distance function \( d(x, y) \), show that \( S \) with the distance function \( \delta(x, y) = \frac{d(x, y)}{1+d(x, y)} \) is also a metric space.

First we see if \( \delta(x, y) = 0 \), then \( \frac{d(x, y)}{1+d(x, y)} = 0 \), and so \( d(x, y) = 0 \), and \( x = y \).

Next, if \( x = y \), then
\[
\delta(x, y) = \frac{d(x, y)}{1+d(x, y)} = \frac{0}{1+0} = 0.
\]

Finally, since \( d(x, y) \) is a distance function, we have
\[
\begin{align*}
d(x, z) & \leq d(x, y) + d(y, z) \\
& \leq d(x, y) + d(y, z) + 2d(y, z) \cdot d(x, y) \\
& \leq d(x, y) + d(y, z) + 2d(y, z)d(x, y) \\
& \leq d(x, y) + d(y, z) + d([x, y])d(y, z) \\
& \leq d(x, y)(1 + d(x, z) + d(x, z)) + d(y, z) \\
& \leq d(x, y)(1 + d(x, z) + d(x, z)) + d(y, z)
\end{align*}
\]

Thus, the new distance function satisfies the triangle inequality, and \( S \) is a metric space.
3.1.2 #7 Show the accumulation points of any set form a closed set.

Consider a set $X$, its closure $X^-$ and the set of accumulation points $X^A$. If we say $X^I$ are the isolated points of $X$, the complement of the set of accumulation points is

$$(X^A)^C = (X^- \setminus X^O)^C$$

which is the set of points $x \notin X^-$ or $x \in X^I$. If $x \in (X^-)^C$, which is open, so $x$ is an interior point of the complement of $X^A$. If $x$ is isolated in $X$, then there is a neighborhood of $x$ whose intersection with $X$ is just $x$. Therefore, $x$ is in the interior of the complement of $X^A$. Thus, the complement of the set of accumulation points is open, and the set of accumulation points is closed.

3.1.3 #3 Prove that the closure of a connected set is connected.

Consider a connected set $X$ and assume there exist $A, B \subset X^-$ such that $A \cap B = \emptyset$ and $X^- = A \cup B$. Then there are disjoint sets $C = A \cap X$ and $D = B \cap X$ such that $C \cup D = X$. Since $X$ is connected, then either $C$ or $D$ must be empty, so that one of their closures is empty. Therefore, $X^-$ must be connected.

3.1.3 #4 Let $A$ be the set of points $(x, y) \in \mathbb{R}^2$ with $x = 0$, $|y| \leq 1$, and let $B$ be the set with $x > 0$, $y = \sin \left( \frac{x}{2} \right)$. Is $A \cup B$ connected?

Since $y = \sin \left( \frac{x}{2} \right)$ is continuous when $x > 0$, it is connected. We can see that any point in $A$ is a limit point of $B$, since any neighborhood of point in $A$ contains an infinite number of points in $B$. Thus, the closure of $B$ is $A \cup B$. By the previous problem, the closure of a connected set is connected, so $A \cup B$ is connected.

3.1.4 #3 Use compactness to prove that a closed bounded set of real numbers has a maximum.

Let $X$ be closed and bounded in $\mathbb{R}$ so that $X$ is compact. Assume $\sup X = x$ is not the maximum of $X$, and define an open cover of $X$ by

$$U = \bigcup_{n \in \mathbb{N}} \left\{ \left( -\infty, x - \frac{1}{n} \right) \right\}.$$ 

Then for any $x_0 \in X$, there is some $n_0$ such that $x - \frac{1}{n_0} > x_0$, and this is in fact a cover. However, any finite subset of $U$ will have a maximum $N \in \mathbb{N}$ so that there is some $x_0 \in X$ such that $x_0 > x - \frac{1}{N}$, and $x_0$ is not covered. That is, there is no finite subcover, contradicting the compactness of $X$. Thus, there must be a maximum of every compact set in the real numbers.
3.1.4 #4 If \( E_1 \supset E_2 \supset E_3 \supset \ldots \) is a decreasing sequence of nonempty compact sets, then the intersection \( \bigcap_{n=1}^{\infty} E_n \) is not empty. Show by example that this need not be true if the sets are merely closed.

We know the arbitrary intersection of closed sets of closed, so \( \bigcap_{n=1}^{\infty} E_n \) is closed. Furthermore, \( \bigcap_{n=1}^{\infty} E_n \subset E_1 \) since the sequence is decreasing, so the intersection is bounded, and thus compact. Now consider a sequence \( \{x_j\} \) where \( x_j \in A_j \) for each \( n \). Then there exists a convergent subcover with its limit in \( A \). In fact, if we remove the first \( k \) sets, we will see that there is a convergent subsequence with limit in \( A_{k+1} \). That is, the limit of the subsequence is in all \( A_j \), and must also be in the intersection. Therefore, the intersection is nonempty.

3.1.5 #1 Construct a topological mapping of the open disk \( |z| < 1 \) onto the whole plane.

Consider a mapping \( f : \{ z : |z| < 1 \} \to \mathbb{C} \) given by \( f(z) = \frac{z}{1 - |z|^2} \). Note that this mapping is continuous where \( |z| \neq 1 \), so that it is continuous on our domain. To see the map is injective, let \( f(z) = f(w) \). Then we have

\[
\frac{z}{1 - |z|^2} = \frac{w}{1 - |w|^2}
\]

which is equivalent to

\[
\frac{z}{w} = \frac{1 - |z|^2}{1 - |w|^2}.
\]

Since \( \frac{1 - |z|^2}{1 - |w|^2} \) is purely real, it must be that \( z = xw \) for some \( x \in \mathbb{R} \). In that case,

\[
\frac{z}{1 - |z|^2} = \frac{xz}{1 - |xz|^2} = \frac{xz}{1 - x|z|^2}.
\]

Rearranging, we get

\[
x - x|z| = 1 - x|z|
\]

which is only true if \( x = 1 \). That is, \( w = z \), so that \( f \) is injective.

To show surjectivity, note that any ball \( |z| < \epsilon < 1 \) maps to a ball \( |w| < \frac{1}{1 - \epsilon^2} \). Thus, taking a limit as \( \epsilon \to 1 \), we get \( |w| \to \infty \), so the function in surjective. Thus we have a bijective continuous function, and it is a homeomorphism.

3.1.5 #3 Prove that every continuous one-to-one mapping of a compact space is topological.

Assume a mapping \( f \) of a compact (metric) space \( X \) is continuous and one-to-one. Then the image of \( f \) is also compact. If we consider a closed subset of \( X \), then it must also be compact, so its image is compact. However, every compact subset of a compact set in a metric space is closed, so the image is closed. That is, \( f \) maps closed sets to closed sets, which means the inverse image of \( f \) is continuous. Furthermore, since \( f \) is one-to-one onto its image, the inverse is also one-to-one, so \( f \) is topological.
3.2.2 #1 Give a precise definition of a single-valued branch of $\sqrt{1+z} + \sqrt{1-z}$ in a suitable region, and prove that it is analytic.

First begin by noting the principal branch of $\sqrt{z}$ is given by $\mathbb{C} \setminus (0, -\infty)$. Then shifting the values of $z$ we find that $\mathbb{C} \setminus (\infty, -1)$ is a single-valued branch of $\sqrt{1+z}$ and $\mathbb{C} \setminus (1, \infty)$ is a single valued branch of $\sqrt{1-z}$. Taking the intersection, so that both functions are single-valued, a single valued branch cut for $\sqrt{1+z} + \sqrt{1-z}$ is $\mathbb{C} \setminus \{(-\infty, -1), [1, \infty]\}$.

The derivative is given by $\frac{1}{\sqrt{1+z}} - \frac{1}{\sqrt{1-z}}$. Then the derivative is undefined at $z = \pm 1$, but those are excluded from our domain. In addition, we know $\sqrt{z}$ is analytic, and sums of analytic functions are again analytic, so $\sqrt{z}$ is given by $\mathbb{C} \setminus (-\infty, 0]$ is analytic.

3.2.2 #2 Give a precise definition of a single-valued branch of $\log(\log z)$ in a suitable region, and prove that it is analytic.

The principal branch of $\log z$ is given by $\mathbb{C} \setminus (-\infty, 0]$. Its image is the slit plane $\{z = x + iy : |y| < \pi\}$. Removing the interval $(-\infty, 0]$ from the image would give another single-valued function, and the inverse image of $(-\infty, 0]$ under $\log z$ is $(0, 1]$. Therefore, a single-valued branch of $\log(\log z)$ is $\mathbb{C} \setminus (-\infty, 1]$.

Its derivative is $\frac{1}{z \log z}$, which is undefined at $z = 0, 1$, which are not in our domain. Furthermore, $\log z$ is analytic and the composition of analytic functions is analytic, so $\log(\log z)$ is analytic.