All the exercises are obtained from *Complex Analysis*, Third Edition, by Lars Ahlfors.

1. (Exercise 1, Section 4.2.3) Compute

(a) \( \int_{|z|=1} e^z z^{-n} \, dz \);

(b) \( \int_{|z|=2} z^n (1 - z)^m \, dz \);

(c) \( \int_{|z|=\rho} |z - a|^{-4} \, dz \), where \( |a| \neq \rho \).

**Solution.**

(a) Since \( |z| = 1 \) is a circle containing 0, we know that

\[
\int_{|z|=1} \frac{e^z}{z^n} \, dz = \frac{2\pi i}{(n-1)!} \left( \frac{(n-1)!}{2\pi i} \int_{|z|=1} \frac{e^z}{(z-0)^n} \, dz \right)
\]

\[
= \frac{2\pi i}{(n-1)!} \left. \frac{d^{n-1}}{dz^{n-1}}(e^z) \right|_{z=0}
\]

\[
= \frac{2\pi i}{(n-1)!}
\]

We have implicitly assumed that \( n > 0 \). If \( n \leq 0 \), then the exercise is trivial for \( e^z z^{-n} \) would be an analytic function — so the integral would evaluate to 0.

(b) We consider two separate cases. We do not treat the case when \( n, m \geq 0 \) for the integrand \( z^n (1 - z)^m \) is analytic in the inside of \( |z| = 2 \), so the value of the integral is 0.

i. Case \( n < 0, m \geq 0 \): We see that

\[
\int_{|z|=2} \frac{(1 - z)^m}{z^n} \, dz = \frac{(n-1)!}{2\pi i} \left( \frac{2\pi i}{(n-1)!} \int_{|z|=1} \frac{(1 - z)^m}{(z-0)^n} \, dz \right)
\]

\[
= \frac{2\pi i}{(n-1)!} \left. \frac{d^{n-1}}{dz^{n-1}}(1 - z)^m \right|_{z=0}
\]

\[
= \frac{2\pi i}{(n-1)!} \left. \frac{d^{n-1}}{dz^{n-1}} \sum_{r=0}^{m} \binom{m}{r} (-1)^r z^r \right|_{z=0}
\]

\[
= \left\{ \begin{array}{ll}
0, & \text{if } m > n - 1, \\
(-1)^{n-1} \frac{2\pi i}{(n-1)!} \frac{m}{(n-1)(n-2) \cdots 2 \cdot 1}, & \text{if } m \leq n - 1,
\end{array} \right.
\]
or, in other words,
\[
\int_{|z|=2} \frac{(1-z)^m}{z^n} \, dz = \begin{cases} 
0, & \text{if } m > n - 1, \\
(-1)^{n-1} 2\pi i \binom{m}{n-1}, & \text{if } m \leq n - 1,
\end{cases}
\]

ii. Case \( n \geq 0, m < 0 \): Very similar to the previous case. We have
\[
\int_{|z|=2} \frac{z^n}{(1-z)^m} \, dz = \frac{2\pi i}{(-1)^m (m-1)!} \left( \frac{(m-1)!}{2\pi i} \int_{|z|=2} \frac{z^n}{(z-1)^m} \, dz \right)
= (-1)^m \frac{2\pi i}{(m-1)!} \left. \frac{\partial^{m-1}}{\partial z^{m-1}} z^n \right|_{z=1}
= \begin{cases} 
0, & \text{if } m - 1 > n; \\
(-1)^m \frac{2\pi i}{(m-1)!} n(n-1) \cdots (n-m+1), & \text{if } m - 1 \leq n.
\end{cases}
\]

(c) Recall that \( |dz| = i\rho \frac{dz}{z} \). First notice that if \( a = 0 \), then the given integral reduces to
\[
-i\rho \int_C \frac{dz}{\rho^3 z} = \frac{2\pi}{\rho^3}.
\]
Now we focus on the case when \( a \neq 0 \). Specifically, we will consider the cases where \( |a| > \rho \), and \( |a| < \rho \). After much algebra, we obtain
\[
\int_C \frac{|dz|}{|z-a|^4} = -i\rho \int_C \frac{dz}{(z-\frac{a}{\rho})^2 (z-a)^2}.
\]
Let \( b = \frac{a}{\rho} \). In the former case, we have that \( z/(z-a)^2 \) is analytic in \( C \). Therefore,
\[
\frac{2\pi \rho}{a^2} \left( \frac{a + b}{(a-b)^2} \right).
\]
The result turns out to be identical for \( |a| < \rho \).

2. (Exercise 2, Section 4.2.3) Prove that a function that is analytic in the whole plane and satisfies an inequality \( |f(z)| < |z|^n \) for some \( n \) and all sufficiently large \( |z| \), reduces to a polynomial.

Proof. We can make the following estimate of the \( k \)th derivative of \( f \) at a point \( z \):
\[
|f^{(k)}(z)| \leq \frac{n!}{2\pi} \int_C \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} \, |d\zeta|
= n! \frac{|z|^n}{|z|^k}
= n! |z|^{n-k}.
\]
Setting \( k = n + 1 \), we have

\[
|f^{(n+1)}(z)| \leq \frac{(n+1)!}{|z|} \to 0
\]
as \( |z| \to \infty \). Since the above assertion holds for any \( z \), we conclude that \( f \) must be a polynomial.

3. (Exercise 3, Section 4.2.3) If \( f \) is analytic and \( |f(z)| \leq M \) for \( |z| \leq R \), find an upper bound for \( |f^{(n)}(z)| \) in \( |z| \leq \rho < R \).

Solution. Instead of using \( r \) in Cauchy's estimate, which corresponds to the radius of the circle centered at some point, we can now surround \( z \) (the point at which we evaluate the derivative \( f^{(n)} \)) by a circle of radius at most \( R - \rho \). Therefore, we have the upper bound

\[
|f^{(n)}(z)| \leq \frac{n!M}{(R - \rho)^n}
\]

4. (Exercise 5, Section 4.2.3) Show that the successive derivatives of an analytic function at a point can never satisfy \( |f^{(n)}(z)| > n!n^n \). Formulate a sharper theorem of the same kind.

Proof. Let \( C \) be a circle of radius \( r \) centered at \( z \) such that \( f \) is analytic inside \( C \) (and on \( C \)). For the \( n \)th derivative of \( f \), we know the estimate

\[
|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_C \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} \, d\zeta \\
\leq \frac{n!M(2\pi r)}{(2\pi)^{n+1}r^{n+1}} \\
\leq \frac{n!M}{r^n},
\]

where we define \( M \) to be \( \sup |f(\zeta)| \), which we know must be finite since we are taking the supremum of \( f \) over a compact set. Now let \( n \) be \( \max\{1, M/r\} \), so we have \( nr \geq M > 1 \), which implies that \( (nr)^n \geq M \Rightarrow n^n \geq \frac{M}{r^n} \).

Substituting the above inequality into our estimate for the \( n \)th derivative of \( f \) yields \( |f^{(n)}(z)| \leq n!n^n \), which implies that the derivative of \( f \) never satisfies the strict inequality \( |f^{(n)}(z)| > n!n^n \).