Theorem: If \( f \) analytic on \( D - \bigg\{ \sum_{j=1}^{k} S_j \bigg\} \) such that

\[
\lim_{z \to S_j} (z - S_j)f(z) = 0 \quad \forall \ j = 1, \ldots, k
\]

then

\[
\oint_{\gamma} f(z) \, dz = 0
\]

for any (smooth) closed curve \( \gamma \subseteq D - \bigg\{ \sum_{j=1}^{k} S_j \bigg\} \).

To combine earlier results, proved true on disk \( D \), true for a rectangle

\( R - \bigg\{ \sum_{j=1}^{k} S_j \bigg\} \), for

finite \# of pts. \( S_j \).

Just choose rectilinear paths in \( D \)

to avoid \( S_j \)’s:

Complete proof as before.

---

Key ingredient in pf. of Cauchy's integral formula: Suppose \( f \) analytic on \( D \).

Let \( \phi(z) = \frac{f(z) - f(a)}{z - a} \).

Then

\[
\lim_{z \to a} \phi(z) \cdot (z - a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = 0 \quad \text{(since } f \text{ analytic implies } f \text{ continuous)}
\]

Hence:

\[
\oint_{\gamma} \left( \frac{f(z) - f(a)}{z - a} \right) \, dz = 0
\]

or equivalently:

\[
\oint_{\gamma} \frac{f(z)}{z - a} \, dz = f(a) \cdot \oint_{\gamma} \frac{1}{z - a} \, dz
\]

\( \gamma \) is circle centered at \( a \) with sufficiently small radius \( r \).

If \( \gamma = C(a; r) \subseteq D \), then

\[
\oint_{\gamma} \frac{1}{z - a} \, dz = 2\pi i
\]
Conclusion: \[ f(a) = \frac{1}{2\pi i} \oint_{C(a;r)} \frac{f(z)}{z-a} \, dz \]

We can understand the value of the function at any point \( a \) in disk, just by knowing its values on \( C(a;r) \). In fact, similar statement is true for general closed curves in \( \mathbb{C} \). Just need to analyze \( \oint_{\gamma} \frac{dz}{z-a} \).
Believe the circle \( C(a,r) \) centered at \( a \) is fundamental.

Traversing circle twice counter-clockwise gives \( 2 \cdot (2\pi i) \). Expect a curve of shape:

Further, if \( \gamma \) has self-intersections:

then can be separated into multiple piece-wise smooth closed curves. Previous thm \( \Rightarrow \) \( \int_{\gamma_i} \frac{1}{z-a} \, dz = 0 \) if \( \gamma_i \) does not contain \( a \).

Conjecture: \( \int_{\gamma} \frac{dz}{z-a} = n \cdot 2\pi i \), where \( n \) is an integer.

if \( \gamma \) closed curve containing \( a \).

How to prove this? \( 2\pi i \) is period for \( e^z \). If \( \gamma \) parametrized by \( z(t) \), \( t \in [\alpha, \beta] \), then

\[
\int_{\gamma} \frac{dz}{z-a} = \int_{\alpha}^{\beta} \frac{z'(t)}{z(t)-a} \, dt
\]

so consider

\[ h(t) := \int_{\alpha}^{t} \frac{z'(s)}{z(s)-a} \, ds \] is continuous, \( h : \mathbb{R} \to \mathbb{C} \).
with  \( h'(t) = \frac{z'(t)}{z(t) - a} \). Want to show \( e^{h(t)} = 1 \).

\[
\frac{d}{dt} e^{h(t)} = \frac{z'(t)}{z(t) - a} e^{h(t)} , \text{ so } \frac{d}{dt} (z(t) - a) e^{-h(t)} = 0 \text{ for all } t .
\]

\[ \Rightarrow (z(t) - a) e^{-h(t)} \text{ is constant} . \]

If \( t = a \), then \( h(a) = 0 \), so

\[ z(a) - a = (z(t) - a) e^{-h(t)} \]

(e. \( e^{h(t)} = \frac{z(t) - a}{z(a) - a} \)).

Setting \( t = \beta \), we see that the right-hand side is 1. So \( e^{h(\beta)} = 1 \).

\[ \Rightarrow h(\beta) = \int_\alpha^\beta \frac{dz}{z(t) - a} \text{ is multiple of } 2\pi i . \]

Define "winding number" to be this integer: \( n(\gamma, a) = \int \frac{dz}{z - a} \cdot \left( \frac{1}{2\pi i} \right) \).

Properties:

1. \( n(\gamma, a) = -n(-\gamma, a) \)

2. \( n(\gamma, a) = 0 \) if \( \gamma \) contained in disk \( D \), \( a \notin D \).

3. \( \gamma \) cuts \( C \cup \Sigma \) into open, connected sets, \( n(\gamma, a) \) constant on a region.
pf of \( \bigcirc \): Any two points \( a, b \in \Omega \) : open connected

are joined by path consisting of straight line segments.

So suffices to examine case when \( a, b \) joined by a single straight line segment.

show \( n(\gamma, a) = n(\gamma, b) \) in this case.

Clever fact: \( \frac{z-a}{z-b} \) is only real, negative on segment connecting \( a \) to \( b \).

\[ \Rightarrow \log \left( \frac{z-a}{z-b} \right) \text{ is well-defined single valued function off the line segment } [a,b]. \]

with derivative \( \frac{1}{z-a} - \frac{1}{z-b} \), so \( \int_{\gamma} \left( \frac{1}{z-a} - \frac{1}{z-b} \right) dz = 0 \)

i.e. \( n(\gamma, a) = n(\gamma, b) \).

To show \( n(\gamma, a) = 0 \) if \( a \) in component containing \( \infty \), already know constant, so

just need to know \( n(\gamma, a) = 0 \) at some point.

a with

Pick \( |a| \) suff. large so that \( \gamma \) contained in a disk away from \( a \).
Thus we have proved the following theorem:

Theorem: Suppose $f$ is analytic on open disk $D$, $\gamma$ closed curve in $D$. Then for any point $a \notin \gamma$, then

$$f(a) = \frac{1}{2\pi i \cdot n(\gamma, a)} \oint_{\gamma} \frac{f(z) \, dz}{z - a}$$

$n(\gamma, a)$: winding number.