On Wednesday, proved Cauchy’s integral formula:

Given \( f \), analytic on disk \( D \), \( \gamma \in D \), a pt. not on \( \gamma \)

then \( f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) \, dz}{z-a} \)

\( n(\gamma, a) \): winding # of \( \gamma \) about \( a \).

If used Cauchy’s thm on disk \(-\) fin. # of pts \(-\), applied to \( \gamma \),

\( \psi(z) = \frac{f(z) - f(a)}{z - a} \).

Cauchy’s int. formula gives us explicit expressions for derivatives as well.

Rewrite integral formula:

\( f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi) \, d\xi}{\xi - z} \)

Then can hope that

\( f'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi) \, d\xi}{(\xi - z)^2} \) \hspace{1cm} \text{(by differentiating integrand w.r.t. parameter \( z \).)}

Indeed if

\( \psi(t, z) = [a, b] \times \Omega \longrightarrow C \)

\( \Omega \): open, \( \psi \) continuous

and analytic as a function of \( z \) for fixed \( t \), with \( \frac{\partial \psi}{\partial z} \) continuous

(i.e. cont. first order partials)

then

\( \int_{a}^{b} \frac{\partial \psi}{\partial z} \, dt =: \psi'(z) \)

where

analytic on \( \Omega \) with derivative \( \psi'(z) = \int_{a}^{b} \frac{\partial \psi}{\partial z} \, dt \).
The proof can be reduced to the real case:

\[ f(t, x) : [a, b] \times [c, d] \text{ continuous.} \quad \ldots \quad \text{same statement.} \]

Estimate difference quotient for \( F(x) \) via mean value thm.

Then use uniform continuity. \( \checkmark \)

Ahlfors has elegant, direct argument for showing one can differentiate w.r.t. parameter under integral sign for just the kind of functions that appear in the Cauchy integral formula.
Lemma: \( \phi(z) \): continuous on arc \( \gamma \). Then

\[
F_n(z) = \int \frac{\phi(s)}{(s - z)^n} ds
\]

defines analytic function in each of the regions determined by \( \gamma \) and

\[
F_n'(z) = n \cdot F_{n+1}(z)
\]

(By induction)

\( \text{if } \ F_i \ \text{continuous:} \)

Given \( z_0 \in \gamma \), find open nbhd

\( B(z_0, \delta) \) such that \( B(z_0, \delta) \subseteq \gamma \), i.e. \( |s - z| > \delta \) if \( s \in \gamma \).

(\( \delta \) as small as we like)

Now \( F_1(z) = F_1(z_0) = \frac{1}{\delta^2} \int_{\gamma} \frac{\phi(s)}{s - z_0} ds \)

\[
\Rightarrow \quad |F_1(z) - F_1(z_0)| < \frac{1}{\delta^2} \int_{\gamma} |\phi(s)| ds
\]

Moreover \( F_1(z) - F_1(z_0) \)

\[
\frac{1}{\delta^2} \int_{\gamma} \frac{\phi(s)}{(s - z)(s - z_0)} ds
\]

\( \Rightarrow \quad \text{fixed const. indep. of } z, z_0. \)

so \( F_1 \) continuous as desired.

\( \therefore \quad F_1'(z) = F_2(z) \)

for all points \( z \) in open, connected \( \gamma \) defined by \( \gamma \).

This is base case of induction.
Suppose now \( F_{n-1}(z) = (n-1)F_n(z) \). We have the slightly messier identity:

\[
F_n(z) - F_n(z_0) = \left[ \int_0^z \frac{\phi(s)ds}{(s-z)^{n-1}(s-z_0)} - \int_0^{z_0} \frac{\phi(s)ds}{(s-z_0)^n} \right] + \int_{z_0}^{z} \frac{\phi(s)ds}{(s-z)^{n}(s-z_0)} \cdot \frac{z-z_0}{z_0-z_0} + \frac{z-z_0}{z_0-z_0} = \frac{z-z_0}{z_0-z_0} = 1.
\]

\[\Rightarrow \text{in continuous, since by inductive hypothesis applied to } \frac{\phi(s)}{s-z_0}, \text{the integrals in brackets are continuous.}\]

The latter integral can be estimated as before:

\[
\text{in abs. value: } < \frac{1}{z_0-\epsilon} \int_{\epsilon}^{1}\frac{1}{s+1}ds.
\]

Similar to before, also divide both sides of identity by \( z-z_0 \).

Take limit as \( z \to z_0 \).

By inductive hypothesis get the term in brackets giving \( (n-1)F_{n+1}(z_0) \).

Since we apply inductive hypothesis to \( F_{n-1}(z_0) \)

\[
F_{n-1}(z) = \int_0^z \frac{\phi(s)}{(s-z)^{n-1}}ds
\]

while latter term gives \( F_{n+1}(z_0) \)

for total of \( n \cdot F_{n+1}(z_0) \).
Apply this to integrals appearing in Cauchy integral formula:

\[ \gamma = \text{circle } C \quad \text{(traversed once, so winding number is one)} \]

Then

\[ \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - \alpha} \, dz = \frac{1}{2\pi i} F_{\gamma}(\alpha) \]

so

\[ f^{(n)}(\alpha) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - \alpha)^{n+1}} \, dz \quad \text{for an analytic function on } \text{interior / exterior of circle } C. \]

Play with this in various ways:

- If \( f \) defined, continuous on \( \Omega \) : open, connected, \( \oint f \, dz = 0 \) for all closed curves \( \gamma \in \Omega \).

\[ \Rightarrow \quad f \text{ is derivative of analytic function } F \text{ earlier than } \]

\[ \Rightarrow \quad f \text{ is analytic.} \quad \text{(Morse's thm)} \]

(Using Cauchy integral theorem on \( F \) and diff.)

Liouville's thm: Function \( f \) which is analytic + bounded in whole plane must be constant.

\[ f : \text{Suppose } C : \text{circle of radius } r, \quad |f(z)| \leq M \text{ on } C \]

then by formula for \( n \)th derivative:

\[ f^{(n)}(\alpha) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - \alpha)^{n+1}} \, dz \]
Then for any \( \alpha \), \( \beta \neq a \) not on circle:

\[
|f^{(n)}(\alpha)| \leq M \cdot n! \cdot r^{-n} \quad (\text{Cauchy's estimate})
\]

\( \text{nevermind the } 2\pi i \).

Use this for \( n=1 \); our hypothesis that \( f \) bounded means \( \exists M \) s.t.

\[
|f(\beta)| \leq M \text{ on any circle } C \ni a, \text{ say containing } a,
\]

hence Cauchy's estimate \( \implies |f'(\alpha)| = 0 \) by taking \( r \to \infty \).

(i.e. \( f'(\alpha) = 0 \))

\( \iff f \text{ constant.} \)

CoV.: (Fundamental Thm. of Algebra)

Every polynomial \( P(\alpha) \), complex has

\( f \): Suppose not. Then \( 1/P(\alpha) \) is analytic,

and bounded since \( |1/P(\alpha)| \to 0 \) as \( \alpha \to \infty \).

\( f: \quad \mathbb{C} \cup \{\infty\} \to \mathbb{R} \) is continuous function, so attains maximum.

But then Liouville's thm implies that \( 1/P(\alpha) \) constant. \( \Box \).