Cauchy's integral theorem:

\[ f(z) \text{ analytic on open disk } D, \quad \gamma \text{ closed curve, } a \notin \gamma \]

then

\[ n(\gamma, a) \cdot f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a} \, dz \]

and we could extend this to \( f \) analytic on \( D - \) finite # of pts?

so long as \( a \neq \xi_j \) some \( j \).

What happens at exceptional points?

\[ \text{thm: } f(z) \text{ analytic on } \Omega \setminus \xi_a \Omega. \text{ Then } f_{21} \text{ may be extended to an analytic function on } \Omega \iff \lim_{z \to a} (z-a) f(z) = 0. \]

(agreeing with \( f \) on \( \Omega \setminus \xi_a \Omega \))

\((\Rightarrow)\) \( f \) analytic at \( a \Rightarrow f \) continuous at \( a \), so \( \lim_{z \to a} f(z) = f(a) \)

and hence \( \lim_{z \to a} (z-a) f(z) = 0. \)

Note: Any such extension is moreover unique, since we require \( f \)
continuous at \( a \), so \( f(a) = \lim_{z \to a} f(z). \)

\((\Leftarrow)\) By Cauchy integral formula (which applies in generalized form)

(\( \text{since } \lim_{z \to a} (z-a) f(z) = 0 \))

then

\[ f(z) = \frac{1}{2\pi i} \oint_{C} \frac{f(z)}{z-a} \, dz \]

where \( C \) suff. small circle about \( a \).
However, last time we proved that, for \( f \) continuous on \( \Omega \)

\[
\int_{\gamma} \frac{f(\xi)}{(\xi - a)} \, d\xi
\]
defines an analytic function for all \( z \) in “regions det'd by \( \gamma \)” interiors of simple closed paths cut out by \( \gamma \).

So RHS of (*) makes sense at \( z = a \).

If we take this to be the newly assigned value with \( \gamma = C \) : circle about a

\[ f(a) := \int_{C} \frac{f(\xi)}{(\xi - a)} \, d\xi \]

the resulting function is analytic on \( C \), agreeing with \( f(z) \) on \( \Omega \setminus \gamma \).

Use this result to do Taylor approximation:

(\( f \) analytic on \( \Omega \))

\[ f(z) = f(a) + (z-a) f_1(z) \quad \text{since} \quad \frac{f(z) - f(a)}{z-a} \]
satisfies conditions of theorem

for some function \( f_1 \) analytic in the region \( \Omega \) containing \( a \).

Recurisely define

\[ f_1(z) = f_1(a) + (z-a) f_2(z) \ldots \]

so

\[ f(z) = f(a) + (z-a) f_1(a) + \cdots + (z-a)^n f_n(z) \]

\((***)\)

after times:

and by differentiating both sides

\[ f_n(a) = \frac{f^{(n)}(a)}{n!} \]

**Note!**

What can we say about remainder term \( f_n(z) \)? Use Cauchy integral formula to estimate it.
\[ f_n(z) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^n(z-z)} \quad C: \text{circle containing } a, z \in C \]

Now \[ f_n(z) = \frac{f(z)}{(z-a)^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{(n-k)! (z-a)^{n-k}} \]

only this term remains in subist.

for any of these terms

\[ \int \frac{1}{j!} \frac{f^{(j)}(a)}{(z-a)^{n-j}} \]

\[ \int \frac{1}{j!} \frac{f^{(j)}(a)}{(z-a)^{n-j}} = 0 \]

by partial fractions.

(similar to how you did last week)

\[ f_n(z) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^n(z-z)} \]

We can use this to prove if \[ f^{(n)}(a) = 0 \quad \forall \quad n \]

then \[ f = 0 \]

on open, conn. set \( \Omega \).

\[ |f(z)| \leq M \quad \text{on our circle } C \quad (\text{compact}) \]

\[ \Rightarrow \quad |f_n(z)| \leq \frac{M}{R^{n-1}} (R-|z-a|) \quad \text{if } C \text{ has radius } R. \]

But if first \( n-1 \) derivatives vanish:

\[ f(z) = f_n(z)(z-a)^n \quad \Rightarrow \quad |f(z)| \leq \left( \frac{|z-a|^n}{R^n} \right)^{\frac{1}{n}} \]

since \( z \in C \), \( |z-a|^n \to 0 \) as \( n \to \infty \)

\[ \text{Hence } f(z) = 0 \quad \text{on } C. \quad (\text{use topological arg to show identically 0 on } \Omega) \]
Topological argument: \( \Omega = E_1 \cup E_2 \), \( E_1 \): pts at which all derivatives of \( f \) vanish. \( E_2 \): pts for which some derivative is non-zero.

Just shown \( E_1 \) open (vanishes on small nbhd containing \( a \)). and \( E_2 \) open since all derivatives are continuous.

But \( \Omega \) connected \( \Rightarrow \) \( E_1 \) or \( E_2 \) empty. \( \Box \)

So if \( f \neq 0 \), \( f(a) = 0 \), then some derivative \( f^{(k)}(a) \neq 0 \) for some \( k \). Smallest such \( k \) is called the order of the zero. Write \( f(a) = (z-a)^k f_k(z) \) with \( f_k(z) \neq 0 \) at \( z = a \).

So zeros of analytic functions are isolated.

In this way, analytic functions are like polynomials.