Plan for the day:
- a few more consequences of C.I.F.
- discuss what's next after midterm
- reminders about content of midterm

Consequences of C.I.F. So far:
- Morera's Thm: \( \text{meromorphic in } \Omega \Rightarrow f \text{ analytic} \) of \( f \)
- Liouville's Thm: \( \text{bounded, entire functions constant} \)
- Cauchy's estimate on \( n^{th} \) derivatives

\( f^n(z) \) is holomorphic on \( \Omega \)

\[ f^n(z) = f(a) + f'(a)(z-a) + \cdots + f^{n-1}(a)(z-a)^{n-1} + \frac{f^n(a)}{(n-1)!} \]

with \( f_n(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-a)^n(s-z)} \, ds \)

Perform Cauchy-ineq estimate on this. \( f \) continuous on \( C \) (compact) means \( |f(z)| \leq M \)

then \( |f_n(z)| \leq \frac{M}{R^{n-1} \cdot \left(R-|z-a|\right)} \)

if \( C \) is compact of \( a \) of radius \( R \).

\( C \) inside \( C \) so \( |z-a| < R \).
Then: if analytic in \( \Omega \), a.e. \( \Omega \) s.t. \( f^{(n)}(a) = 0 \) \( \forall n \geq 0 \)

then \( f \equiv 0 \) in \( \Omega \).

Identically 0 at all pts in \( \Omega \).

If: Since derivatives at \( a \) vanish,

\[ f(z) = (z-a)^n f_n(z) \]

for any \( n \).

By our estimate

\[ |f(z)| \leq \frac{|z-a|^n}{R^n} \cdot \frac{M_R}{R-|z-a|} \]  

Take limit as \( n \to \infty \)

\[ \frac{|z-a|}{R} < 1 \]  

so

\[ |f(z)| \]  

\Rightarrow \( f(z) = 0 \) on interior of \( C \).

Left to show: \( f \) is 0 on all of \( \Omega \).

(\( C \) was just circle centered about \( a \) inside \( \Omega \))

Clearer topological arg.:

\( \Omega = E_1 \cup E_2 \)

pts. \( z_0 \in \Omega \) s.t. all derivs of \( f \) vanish.

\( E_1 \) : all derivs of \( f \) at \( z_0 \) vanish.

\( E_2 \) : pts. \( z_0 \) s.t. some deriv. doesn't vanish.

Previous argument via circles \( \Rightarrow E_1 \) open.

\( E_2 \) open since derivs of \( f \) are continuous, so \( f^{(k)}(z_0) \to 0 \)

say equal to \( z_1 \)

take open nbhd of \( z_1 \) not containing 0, im. image under \( f(z) \) is open.

But \( \Omega \) open, conn. \( \Rightarrow \) either \( E_1, E_2 \) empty.

But \( a \in E_1 \) so

must be \( E_2 \) is empty.

\( \Rightarrow f \equiv 0 \) in \( \Omega \).
Turn this logic around:

If $f \neq 0$ on $\Omega$, then smallest $h$ s.t. $f^{(h)}(a) \neq 0$ at any $a \in \Omega$.

(i.e. order of a zero of $f$ is finite, for all analytic functions)

Write it $f(z) = (z-a)^n f_n(z)$

with $f_n(a) \neq 0$ just as for polynomials.

In fact, $f_n(z) \neq 0$ in nbhd of $a$.

(since $f_n(z)$ analytic, so also continuous)

non-zero

so zeros of analytic function are isolated.

[All or Nothing rule]

Corollary: If $f, g$ analytic on $\Omega$ and $f(z) = g(z)$ for $z \in S$, set

then $f = g$ for all $z \in \Omega$.

pf: For $z \in S$, $f - g = 0$ so zeros of $f - g$ not isolated

$\implies f - g \equiv 0$ on $\Omega$. 