Other use of isolated zeros / Cauchy int. formula:

Suppose $f$ analytic on disk $D$, $\gamma$ : closed curve in $D$ with
with finitely many zeros $z_1, \ldots, z_n$ not on $\gamma$.

Write $f(z) = (z - z_1) \cdots (z - z_n) g(z)$ with $g(z)$ analytic, $\neq 0$ on $D$.

(Here $z_i$'s may not be distinct.

can be repeated according to multiplicity)

Just as in Lucas-Gauss thm. on zeros of polynomials:

take logarithmic derivative to get

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)},$$

Integrate both sides over $\gamma$.

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{n} \left( \text{index of } \gamma \text{ at } z_j \right) \quad \text{(since } \int_{\gamma} \frac{g'(z)}{g(z)} = 0)$$

b/c $g(z) \neq 0$ on $D$

so $g'(z)/g(z)$ is analytic on $D$.

Here use that we only know

Cauchy's thm. on disk $D$.

Note that our assumption about $f$ having finitely
many zeros is unnecessary since only zeros inside $\gamma$
contribute to equality. There are only finitely many of these
since $\gamma \subset B' \subset D$ for some $B'$ with $B' \cap D$. on this

...and cut $\gamma$. Infinitely many zeros would have an accumulation point.
If curve is simple, e.g. circle, then \( n(z_1, z_2) \) will always be 0, 1

so can think of \( \int_{\gamma} \frac{f'(z) \, dz}{f(z) - a} \) as \( n \) of \( z \)'s \( w = f(z) \) inside \( \gamma \).

Slight generalization: Apply to \( f(z) - a \):

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z) \, dz}{f(z) - a} = \sum_{j} n(z_1, z_j(a))
\]

\( z_j(a) \) are pts. for which

\( f(z_j(a)) = a \),

i.e. solns to \( f(z) = a \).

(need that \( f(z) \neq a \) on \( \gamma \), of course)

Slight reformulation: \( f : \gamma \to \mathbb{C} \) closed curve, write \( w = f(z) \) then think of \( \delta \) as lying in \( w \)-plane.

\[
\begin{align*}
n(\delta; a) & \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{\delta} \frac{dw}{w-a} \\
& = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z) \, dz}{f(z)-a} = \sum_{j} n(z_1, z_j(a))
\end{align*}
\]

Since \( n(\delta; a) \) is constant on regions defined by \( \delta \), then

if \( a, b \) in some region:

\[
\sum_{j} n(z_1, z_j(a)) = \sum_{j} n(z_1, z_j(b))
\]

In words, \( f \) takes values \( a, b \) same number of times if \( a \) suff. close to \( b \) inside \( \gamma \).

Thus: \( f(z) \) analytic in nbhd of \( z_0 \), \( f(z_0) = w_0 \), with \( f(z_0) - w_0 \) having 0 of order \( n \) at \( z_0 \).

In part:

\( f(z) \neq w_0 \)

for suff. small \( \epsilon > 0 \), \( \delta > 0 \) s.t. for all \( z \) with \( |z - z_0| < \delta \)

\( f(z) - a \) has \( n \) roots in a disk \( |z - z_0| < \epsilon \).
As a special case for use in theorems, take $a = \text{circle of radius } r$ around single solution to $x$.

\[ \frac{1}{\sum_n} \frac{1}{\sum_n} \]
If \( \epsilon \) is so that \( z_0 \) is only \( 0 \) of \( f(z) \) in \( \mathbb{D} \),

(can be done since zeros isolated). Apply above result with \( \delta = C(z_0, \epsilon) \),

\[ f(z) - w_0 \]

Let \( \delta = f(z) \). By construction \( f(z) \neq w_0 \) on \( \delta \),

so the result applies.

So \( \exists \delta > 0 \) s.t. \( B(w_0, \delta) \cap \delta = \emptyset \). This is desired \( \delta \).

(Note: by picking \( \epsilon \) sufficiently small, we can assume so that \( f(z) = a \) has multiplicity 1.

Corollary: (Open Mapping Thm.) \( \Omega \) open, conn., \( f \) non-const. analytic function on \( \Omega \),

then \( f \) maps open sets in \( \Omega \) to open sets.

(since previous thm. showed \( f(B(z_0, \epsilon)) > B(w_0, \delta) \))

Corollary 2: (Maximum principle) \( f \) analytic, non-const. on \( \Omega \), then

\[ |f(z)| \] has no maximum on \( \Omega \).

if \( \delta > 0 = f(z_0) \), then \( \exists \delta \) \( B(w_0, \delta) \subset f(B(z_0, \epsilon)) \)

So \( \exists \) point \( w \in B(w_0, \delta) \) with \( |w| > |w_0| \). So \( f(z) \) not maximum.

Alternate (positive) formulation:

\( f(z) \): continuous on compact set \( E \), analytic on interior, then \( \max f(z) \) is attained on the boundary of \( E \).

if \( f \) has a max on \( E \), since \( E \) assumed compact. If max occurs at \( z_0 \)
in interior, then \( f \) must be constant on component of \( E \) containing \( z_0 \).

(\( \sigma \) max also attained on boundary if this component \( E \) boundary of \( E \).)