Play around with maximum principle:

On a disk: If we know $|f(z)| \leq M$ for $z \in D$, of radius $R$, then either $f(z) = M$ on $D$ or else $\exists z_0$ s.t. $|f(z_0)| < M$.

Example: $D$: unit disk, $M = 1$, $f(0) = 0$ for convenience.

Then $|f(z)| \leq |z|$, $|f'(0)| \leq 1$.

Moreover, if $|f(z)| = |z|$ for some $z \neq 0$ or $|f'(0)| = 1$, then $f(z) = cz$ with $|c| = 1$.

Idea: Apply max. principle to $f(z)/z$ (removable singularity at $z = 0$)

and define value at $0$ to be $f'(0)$.

For any $z$ on circle $|z| = r < 1$,

this function has modulus $\leq \frac{1}{r}$, so must be $\leq \frac{1}{r}$ on interior.

as $r \to 1$, get $|f(z)/z| \leq 1$. (if equality holds, then $f(z)/z$ must be constant.)

Can sup over up for circles centered at $z_0$ with $f(z_0) = w_0$, radius $R$.

See (36) in Ahlfors.

Useful in homework.

Aside: What can be done using only Cauchy's inequality?
Maximum modulus principle applies to bounded domains, what can be said about functions on arbitrary region $\Omega$ if we know values of function on $\partial \Omega$. (say $|f(z)| \leq M$ on $\partial \Omega$)

Example: $e^{\frac{z}{2}}$ on horizontal strip: $\{z \mid \text{Im}(z) < \frac{\pi}{2}\}$

On boundary, $|e^{\frac{z}{2}}| = |e^{x(cos y + isin y)}| = e^x \cos y$

But $\cos y = 0$ for $y = \pm \frac{\pi}{2}$, so $|e^{\frac{z}{2}}| = 1$ on boundary.

However $e^{\frac{z}{2}} \to \infty$ as $\text{Re}(z) \to \infty$.

Possible modifications: Suppose that on half-strip $\text{w/ } \text{Re}(z) > 0$, then

$$|f(z)| \leq e^{C \cdot \text{Re}(z)}$$

(i.e., for $|z| > a$, we have $f(z) = e^{C \cdot \text{Re}(z)}$)

$0 < C < 1$

then can use maximum principle to prove

in fact $|f(z)| \leq M$ $\forall z$ in half-strip.

<< See Garrett vignette for proof.

Other possible modifications:

For unbounded domains, $\partial \Omega = \text{finite boundary} \cup \partial \Omega$.

Require $f(z)$ bounded on $\partial \Omega$, which is not satisfied by $f(z) = e^{\frac{z}{2}}$.

That is, require

$$\lim_{z \to a} |f(z)| \leq M$$

for all $a \in \partial \Omega$.

That is, require

$$\lim_{r \to 0} \sup_{z \in \Omega \cap B(a,r)} |f(z)|$$
Let's prove this latter modification for unbounded domains:

**Thm:** \( \Omega : \text{open, conn.} \), \( f : \text{analytic on } \Omega \) with \( \lim_{z \to a} |f(z)| \leq M \) on \( \partial_0 \Omega \).

Then \( |f(z)| \leq M \) for all \( z \in \Omega \).

**Pf:** Given any \( \delta > 0 \). Show \( \overline{\Omega} := \{ z \in \Omega \mid |f(z)| > M + \delta \} \) is empty.

Since \( |f(z)| \) is continuous, then \( \overline{\Omega} \) is open.

Further since \( \lim_{z \to a} |f(z)| \leq M \) on \( \partial_0 \Omega \), then \( \exists \ B(a, r) \) s.t.

\( |f(z)| < M + \delta \quad \forall z \in \Omega \cap B(a, r). \) \( \Rightarrow \overline{\Omega} \subset \Omega \).

\( \Rightarrow \) \( \overline{\Omega} \) compact.

Apply Maximum modulus principle to \( \overline{\Omega} \).

Then noting \( \overline{\Omega} \) consists of points \( z \) s.t. \( |f(z)| = M + \delta \)

b/c \( \overline{\Omega} \) is \( \{ z \mid |f(z)| > M + \delta \} \), either \( f(z) \) constant on \( \overline{\Omega} \)

or \( \overline{\Omega} \) empty.

Even if \( f \) constant, \( \overline{\Omega} \) is empty.

Since \( \lim_{z \to a} |f(z)| \leq M \) on \( \partial_0 \Omega \).