Today: Geometry of complex numbers.

Last time, noted that the distance function was given by:

\[ |z|^2 = \bar{z}z = x^2 + y^2 \quad \text{if} \quad z = x + iy \]

so distance \( d(z, w) := |z - w| \)

where \[ |z - w|^2 = |z|^2 + |w|^2 - 2 \text{Re} \ z \bar{w} \]

Similarly, \[ |z + w|^2 = |z|^2 + |w|^2 + 2 \text{Re} \ z \bar{w}. \quad (*) \]

Want to confirm we have metric, i.e. \( d(z, w) \) satisfies

1. \( d(z, w) \geq 0 \) and \( d(z, w) = 0 \) iff \( z = w \).

2. \( d(z, w) = d(w, z) \) \( \forall \ z, w \in \mathbb{C} \)

3. \( d(z, w) \leq d(z, u) + d(u, w) \) (triangle inequality).

\((3) \) is intuitively clear geometrically, since \( d(z, w) \) is distance between points \( z, w \) in IR^2. But should verify this analytically or algebraically.

Proof of (3): Use (\( *) \), noting that

\[ |\text{Re}(z)| \leq |z| \quad \text{and} \quad |z \bar{w}| = |z| |\bar{w}| = |z||w| \]

So indeed \[ |z + w|^2 \leq |z|^2 + |w|^2 + 2|z||w| \quad \checkmark \]

Note used \( \text{Re}(z \bar{w}) \leq |z \bar{w}| \) so get equality if and only if \( z \bar{w} \) is non-neg. real.

\( \Rightarrow \) \( z, w \) non-neg. real.
so \((C, l_1)\) is metric space (just \(R^2\) with Euclidean norm)

Two other inequalities of note:

1. \[|z - w| \geq |l - w|\]

**If:**
\[|z| = |(z - w) + w| \leq |z - w| + |w| \Rightarrow |z| - |w| \leq |z - w|\]

Play same game with \(|w|\) to get
\[|w| - |z| \leq |w - z| = |z - w|\]  

2. Cauchy’s inequality:
\[
\sum_{i=1}^{n} |z_i w_i|^2 \leq \sum_{i=1}^{n} |z_i|^2 \sum_{i=1}^{n} |w_i|^2
\]

**If:** Introduce parameter \(\lambda\)
\[
\sum_{i=1}^{n} |z_i - \lambda \bar{w}_i|^2 = \text{EXPAND VIA (*)} \Rightarrow \text{optimize this expression as function of } \lambda.
\]

Back to metric spaces, consider new model for \(C \cup \{\infty\}\).

Often functions take the value \(\infty\), it would be nice to be able to pose questions about continuity at \(\infty\), etc. and to place this value on equal footing with complex numbers.

Not a field, but rather set with additional rules:
\[
z \cdot \infty = \infty \quad \frac{z}{0} = \infty \text{ if } z \neq 0
\]
\[
z + \infty = \infty \quad \frac{z}{\infty} = 0 \text{ (if } z \neq 0)\]
Geometric model for $\mathbb{C} \cup \{\infty\}$ — unit sphere in $\mathbb{R}^3$ (Riemann Sphere)

$$S : \{ (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1 \}$$

and identify $\mathbb{C}$ with $(x_1,x_2,0)$.

Idea: points $(x_1,x_2,0)$ in $\mathbb{C}$ correspond to $z = (a_1,a_2,a_3)$ on $S$ where $z$ lies on line connecting $z = (x_1,x_2,0)$ and $(0,0,1)$: north pole of $S$

More explicitly, (because we want algebraic/analytic perspective, not just geometric intuition)

Line connecting $z = (x_1, x_2, 0)$ and $(0,0,1)$ given by parametric equation:

$$\{ [(1-t)x_1, (1-t)x_2, t) \mid t \in \mathbb{R} \} \quad (**)$$

want point on line s.t.

$$(1-t)^2(x_1^2 + x_2^2) + t^2 = 1 \quad \text{(i.e. on} \ S)$$

$$(1-t)^2 |z|^2 = 1 - t^2$$

Solve for $t$. Since $z \neq \infty$, $t \neq 1$, cancel $(1-t)$'s to get

$$t = \frac{|z|^2 - 1}{|z|^2 + 1}$$
If you substitute back into (**) , then

\[ z = (a_1, a_2, a_3) = \left( \frac{2x_1}{|z|^2 + 1}, \frac{2x_2}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \]  

\[ x_1 = \Re(z), \quad x_2 = \Im(z) \]

In reverse direction, given

\[ z = (a_1, a_2, a_3), \]  

then \( t = a_3 \) and can again use (**) 

to get \( x_1 = \frac{a_1}{1-a_3}, \quad x_2 = \frac{a_2}{1-a_3}. \)

so indeed we have a bijection 

\[ \mathbb{C} \cup \{0\} \]  

and 

\[ S. \]

As our distance function, we take Euclidean distance

in \( \mathbb{R}^3 \) from points \( Z, Z' \) on \( S \), corresponding to

\( Z, Z' \in \mathbb{C} \cup \{0\}; \)

and

"chordal distance"

( defines same open sets as usual distance)

Again, explicitly,

\[ d(z, z')^2 = (a_1 - a_1')^2 + (a_2 - a_2')^2 + (a_3 - a_3')^2 \]

\[ = 2 \left( a_1 a_1' + a_2 a_2' + a_3 a_3' \right) \]

since \( Z, Z' \) on \( S \)

Now substitute (***) , to get:

\[ d(z, z') = 2 \frac{|z - z'|}{\sqrt{\left( 1 + |z|^2 \right) \left( 1 + |z'|^2 \right)}} \]

\( Z, Z' \in \mathbb{C} \)

Note: this distance function satisfies triangle inequality since we know same is true of Euclidean distance in \( \mathbb{R}^3 \).

Advantage of Riemann sphere: compact
So far, have rectangular coordinate model (addition corresponds to vector addition) + Riemann sphere (allows us to consider \( \cos \theta \) as value taken by function).

Missing a good model for multiplication: Polar coordinates.

\[ z = x + iy = r (\cos \theta + i \sin \theta) \quad r > 0 \]

(Representation not unique since \((r, \theta)\) and \((r, \theta + 2\pi k)\) \(k \in \mathbb{Z}\) represent same point.)

with \( |z| = r \) since \( \cos \theta + i \sin \theta \) on complex unit circle.

**Maple:** \[ z_1 z_2 = r_1 r_2 \left[ (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\cos \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1) \right] \]

Call \( \theta \) the "argument" of \( z \), and write \( \text{arg}(z) = \theta \).

Then \( \text{arg}(z_1 z_2) = \text{arg}(z_1) + \text{arg}(z_2) \).

Gives geometric interpretation as homothety: rotation + dilation (by \( \text{arg}(z) \)) (by \( |z| \)).

Polar representation gives simple way to solve monomial equations:

\[ z^n = a \quad \text{write} \quad a = r (\cos \phi + i \sin \phi) \]

\[ r = e^{i/n}, \quad \theta = \frac{\phi}{n} + 2\pi k/n \]

(gives \( n \) distinct solutions)