Last time, proved residue theorem:

\[ \text{If analytic function except for isolated singularities in } \Omega : \text{ open, conn.} \]
\[ \forall y \in \Omega, \exists a_j \neq b_j \]

Then,
\[ \frac{1}{2\pi i} \oint \frac{f(z)}{z} \, dz = \sum \frac{n(y, a_j) R_j}{\text{residue of } f \text{ at } a_j} \] (no conv. issue in sum? since \( n(y, a_j) = 0 \) for almost all \( j \)).

Example:
\[ \oint \frac{e^{2z}}{(z-1)^3} \, dz = n(y, 1) R_1 \cdot (2\pi i) \]

\( y \): any smooth closed curve.

Only singularity is pole of order 3 at \( z = 1 \).

\( n(y, 1) \) depends on curve \( y \), but we can compute \( R_1 \).

Formula for residue:
\[ a_{-1} = \frac{1}{(3-1)!} \frac{d^2}{dz^2} \frac{e^{2z}}{(z-1)^3} \bigg|_{z=1} \]
\[ \left( \text{i.e. } \frac{1}{(3-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[ (z-a_j)^k f(z) \right] \bigg|_{z=a_j} \right) \]
\[ = 4e^2 \cdot \frac{1}{(2-1)!} = 2e^2. \]

So answer:
\[ 4\pi i \cdot e^2 \cdot n(y, 1). \]

(HW. p. 154 #112 postponed until next Friday)
If \( f \) has a simple pole (i.e., pole of order 1) then

\[
\alpha_{-1} = \lim_{z \to a_j} (z-a_j) f(z)
\]

or any \( g, h \) analytic except for...

In particular if \( f \) rational function \( f = \frac{g}{h} \), then

\[
\lim_{z \to a_j} (z-a_j) \frac{g(z)}{h(z)} = \frac{g(a_j)}{h'(a_j)}
\]

\[g(a_j)\]

L'Hôpital's rule:

\[
\frac{1}{h'(a_j)}
\]

Example: \( f(z) = \frac{\sin z}{e^{z-3}} \) then \( \alpha_{-1}(f) \) in nbhd of 3 is

\[\sin 3 = \frac{\sin 3}{e^{2 \cdot 3}}\]

since \( \frac{d}{dz} \left( e^{2z} (z-3) \right) = 2e^{2z} \cdot (z-3) + e^{2z} \)

(also see this by thinking about power series expansions...)

Return to our discussion of power series / Laurent series
Power series: if analytic on $B(a,R)$. Pick $r < R$, $\gamma = \text{reit}$

Fix $z \in B(a,r)$. $t \in [0,2\pi]$

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\xi)}{\xi - z} \, d\xi$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - a)^{n+1}} \, d\xi \quad (z-a)^n$$

$$f^{(n)}(a)/n!$$

Play games with fractions:

$$\frac{1}{\xi - z} = \frac{1}{(\xi - a)} \cdot \frac{1}{1 - \left(\frac{z-a}{\xi - a}\right)} = \frac{1}{(\xi - a)} \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\xi - a)^n}$$

Formally, we're done, but need to justify interchange of integration/summation.

$$F_N = \sum_{n=0}^{N} \frac{f(\xi)}{(\xi - a)^n} \quad z,a \text{ fixed.}$$

Show $\int_N \lim_{N} F_N = \lim_{N} \int_N F_N$ if $F_N \to F$ uniformly for all $\xi \in \gamma$.\[\gamma\]

Prove $F_N \to F$ uniformly for all $\xi \in \gamma$.

Second bullet is just:

$$\left| \frac{f(\xi)}{(\xi - a)^n} \frac{(z-a)^n}{(\xi - a)^n} \right| \leq \frac{M}{r} \frac{|(z-a)|^n}{r^n}$$

(Weierstrass M-test)

$|z-a| < 1$ if $z \in B(a,r)$
Laurent series \((\text{first principles})\) \(\sum_{n=0}^{\infty} z^n \mid n = 0, 1, \pm 1, \pm 2, \ldots\) doubly infinite sequence

we say \(\sum_{n=0}^{\infty} z^n\) is absolutely convergent if \(\sum_{n=0}^{\infty} z^n, \sum_{n=1}^{\infty} z^{-n}\) are both absolutely convergent

and then define \(\sum_{n=0}^{\infty} z^n \coloneqq \sum_{n=0}^{\infty} z^n + \sum_{n=1}^{\infty} z^{-n}\)

(similarly, say convergence is uniform if both pieces converge uniformly on a set \(S\)).

want to focus on absolutely convergent series here.

Thm: if analytic on annulus centered at \(z_0\) with radii \(0 \leq R_1 < R_2\)

\[
\text{Set } a_n = \frac{1}{2\pi i} \int_{C(z_0,r)} \frac{f(z)}{(z-z_0)^{n+1}} \, dz
\]

\(r \in (R_1, R_2)\)

Then \(f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n\)

and this series rep in is unique.

\(\check{\text{see p. 184 of Allfors.}}\)

pf sketch: Given \(z \in \text{Ann}(z_0, R_1, R_2)\)

then find \(r_1, r_2\) with \(R_1 < r_1 < r_2 < R_2\) s.t. \(z \in \text{Ann}(z_0, r_1, r_2)\)

Consider cycle \(\gamma = C(z_0, r_2) - C(z_0, r_1) \sim 0\) in \(\text{Ann}(z_0, R_1, R_2)\)
\[ f(z) = \frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - z} \, d\xi \]
\[ = \frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - z} \, d\xi \quad C(2_{1}, 2) \wedge f_2(z) \]
\[ - \frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - z} \, d\xi \quad C(2_{0}, 1) \wedge f_1(z) \]

Compute power series for \( f_1, f_2 \), add them together.

Recall that \( f_2(z) \) defines an analytic function for \( |z - z_0| < r_2 < R_2 \).

But \( r_2 \) arbitrary, so \( f_2(z) \) defines

(Cauchy's theorem says integrals over \( r_2, r_2' \) are equal for any \( R_1 < r_2 < r_2' < R_2 \)*)

analytic function on \( B(2_0, R_2) \).

(\( f_1(z) \) has power series expansion in positive powers)

Similarly, \( f_1(z) \) defines analytic function for \( z \) with \( |z - z_0| > r_1 > R_1 \) so for any \( z \) with \( |z - z_0| > R_1 \) since \( r_1 \) arbitrary again.

\( z \mapsto z_0 + \frac{1}{2} \xi' \), and we're done...

* as long as \( z \not\in C(2_0, r_2) \) or \( C(2_0, r_2') \)