Gamma function (king of all special functions)

working definition: functions that appear often. Just as useless as notion: "elementary function"

Better to try to characterize them.

Euler: want to get interpolate the discretely valued function n! to a continuous real-valued function. Use power series or integral rep.:

$$\int_0^\infty e^{-t} t^n dt = n! \quad (\text{integration by parts n times})$$

So use:

$$\int_0^\infty e^{-t} t^{-s} dt$$

(why this funny normalization? discuss Haar measure for topological groups)

$$\Gamma(s)^{-1} \quad \text{with } \Gamma(n+1) = n!$$

expression defines cx. analytic function for Re(s) > 0.

$$\Gamma(s+1) = s \cdot \Gamma(s)$$

Since, in general by parts,

--

Play around with identities for \( \Gamma(s) \).

$$\Gamma(s) \Gamma(1-s) = \int \int \ldots \quad (\text{makes sense for } \text{Re}(s) \in (0,1))$$

Integrand \( \sim \frac{\text{Re}(s)-1}{t \cdot \text{Re}(s)} \)

whose anti-deriv is \( \frac{\pi}{\sin \pi s} \)

Restriction comes from nbhd of 0.

Interchange order of integration - evaluate inner integral via change of vars

$$\ldots \quad \Gamma(s) \Gamma(1-s) = \int_0^\infty \frac{t^{-s}}{1+t} dt = \int_0^\infty \frac{t^{-s}}{1+t} dt = \frac{\pi}{\sin \frac{\pi}{2}}$$

Our earlier result from contour integration / res. thm.

$$\Gamma\left(\frac{1}{2}\right)^2 = \frac{\pi}{\sin \frac{\pi}{2}} \quad \text{i.e. } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$
Details of interchange of integration:

\[
I^n(0) I^n(1-s) = \int_{\mathbb{R}_0^+} \int_{\mathbb{R}_0^+} u^s v^{1-s} e^{-u} e^{-v} \frac{dw}{v} \frac{du}{u}
\]

\[v \mapsto uv\]

\[
= \int_{\mathbb{R}_0^+} \int_{\mathbb{R}_0^+} (uv)^{1-s} e^{-u(1+v)} \frac{dv}{v} \frac{du}{u}
\]

interchange integration

\[u \mapsto u/(1+v)\]

\[
= \int_{\mathbb{R}_0^+} \int_{\mathbb{R}_0^+} \frac{u}{1+v} v^{1-s} e^{-u} \frac{du}{u} \frac{dv}{v}
\]

evaluate inner int: \[
\int_0^{\infty} e^{-u} du = 1
\]

\[
= \int_{\mathbb{R}_0^+} \frac{v^{1-s}}{1+v} \frac{dv}{v}
\]
Where does Gamma appear?

- Functions of exponential decay common in integral transforms / prob. distributions
  so can relate them to Gamma function by change of vars.

- When expressed as infinite product, has interesting set of poles.

- What is volume of n-sphere?
  
  \[
  \begin{align*}
  n = 1 : & \quad 2r \\
  n = 2 : & \quad \pi r^2 \\
  n = 3 : & \quad \frac{4}{3} \pi r^3 \\
  n \text{ arbitrary} : & \quad \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \cdot r^n
  \end{align*}
  \]

  (\frac{1}{2}\text{-integers are related to } \sqrt{\pi},
  
  but other rationals not expected
  
  to be related to known transcendental)

- "Elliptic integrals" - integrals that arise in
  are length of ellipse

\[
\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\Gamma\left(\frac{1}{3}\right)^3}{\sqrt{8} \ \sqrt[3]{16} \ \pi}
\]

- Theory of Riemann zeta function.