Explore properties of Riemann zeta function (analytic continuation / functional equation) and their application (via contour integration) to the prime number theorem.

**Zeta function** \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad C \to C \)

(Weierstrass' Theorem)

General Theorem: \( f_n \) analytic in \( \Omega - n \)

\( f_n(z) \to f \) in \( \Omega \): region for which each \( z \in \Omega \)

has \( z \in \Omega - n \land n \in \mathbb{N} \)

uniformly on every compact subset of \( \Omega \).

Then \( f \) is analytic in \( \Omega \)
(and \( f_n \to f \) uniformly on every compact subset of \( \Omega \))

One of Cauchy's integral formula: Given any \( a \in \Omega \),
pick closed disk contained in \( \Omega \) : \( B(a,r) \).

Then \( f_n(z) = \frac{1}{2\pi i} \oint_{C(a,r)} \frac{f(s)}{s-z} \) for \( z \in B(a,r) \).

Take limit as \( n \to \infty \) uniform convergence gives

Similar formula for derivatives via integral formula.

\( f(z) = \frac{1}{2\pi i} \oint_{C(a,r)} \frac{f(s)}{s-z} \) for \( z \in B(a,r) \).

Some fact we used for existence of power series:

If continuous, \( f = \lim_n f_n \)
then \( \int f \, dz = \lim_n \int f_n \, dz \). cf. Ahlfors p. 36
for analytic continuation: Better to consider
\[ \xi^*(s) = \pi^{-s/2} \Gamma(s/2) \xi(s). \]
\( \Gamma \) is Gamma function

Then \( \xi^*(s) \) extends to meromorphic function of \( \mathbb{C} \)

with simple pole at \( s = 1 \),
satisfying the functional equation:
\[ \xi^*(s) = \xi^*(1-s). \]

**What is \( \xi^* \)-function? Why natural?**

Answer comes from study of functions with prescribed zeros or poles.

Easy enough when # of zeros/poles finite. What if infinite?

Given \( f \) meromorphic on \( \Omega \).

One idea: \( a_i \) pole, then let
\[ P_i \left( \frac{1}{z-a_i} \right) \] be singular part of Laurent expansion of \( f \) at \( z = a_i \).

Consider
\[ f(z) = \sum_{i \in \text{ poles}} P_i \left( \frac{1}{z-a_i} \right) + g(z) \]

remain analytic in \( \Omega \) if only many poles \( \sum P_i \left( \frac{1}{z-a_i} \right) \) may not converge.

**Theorem (Mittag-Leffler)**: \( \xi(s) \) : cx. numbers \( \lim_{i \to 0} |a_i| = \infty \)

and \( P_i \) : polynomials w/o const. term.

Then there are functions \( f \) meromorphic on \( \mathbb{C} \), with poles at \( a_i \),

singular parts \( P_i \).

All such functions can be written in form:
\[ f(z) = \sum_{i} \left[ P_i \left( \frac{1}{z-a_i} \right) - p_i(z) \right] + g(z) \] (4)

\( p_i \) : suitably chosen poly., \( g(z) \) : entire.
If: w.l.o.g., assume none of \( a_i \) are 0. (else shift function)

Then \( P_i \left( \frac{1}{z-a_i} \right) \) is analytic for \( |z| < |a_i| \), so has

Taylor series expansion at origin. Then let \( p_i(z) \) be Taylor polynomial

\[ P_i \left( \frac{1}{z-a_i} \right) = p_i(z) \]

at 0 with remainder

By Cauchy’s inequality: if max of \( |p_i(z)| \) on

\[ |z| \leq |a_i|/2 \]

is given by \( M_i \), then

the remainder

\[ P_i \left( \frac{1}{z-a_i} \right) - p_i(z) = \left( \frac{1}{2\pi i} \right) \oint_C \frac{F(s)}{(s-a_i)^{n+1}} \frac{ds}{s} \]

so

\[ |P_i \left( \frac{1}{z-a_i} \right) - p_i(z)| \leq \frac{1}{2\pi} M_i \left( \frac{2|z|}{|a_i|} \right)^{n+1} \left( \frac{|z|}{|a_i|} \right)^{n+1} \]

where \( C = C(0, |a_i|/2) \) and we restrict \( z \to B(0, |a_i|/4) \).

If we choose \( n_i \) large enough, e.g.,

\[ 2^n_i > M_i \cdot 2^i \]

so that

the RHS of the above inequality will be

\[ 2^n_i \left( \frac{2|z|}{|a_i|} \right)^{n_i+1} \]

and

\[ |z| \leq |a_i|/4 \]

so

\[ z \cdot M_i \cdot 2^{-(n_i+1)} \leq z^{-i} \]

for \( |z| \leq |a_i|/4 \).

Claim: RHS of (‡) in theorem converges uniformly on any disk except at poles,

and hence represents meromorphic function.

Write sum as \( h(z) := \sum_i \frac{1}{z-a_i} - p_i(z) \)

finite sum, \( p_i(z) \) poly.

holom. for \( |z| \leq R \) since

thus our estimate applies.
this proves result since any other if meromorphic with these properies will
have \( f(z) - h(z) \) holomorphic.

Example: \( \frac{\pi^2}{\sin^2 \pi z} \), which has double poles at all integers (and nowhere else in \( \mathbb{C} \))

\[ \lim_{z \to \infty} \frac{\sin \pi z}{z} = \frac{1}{\pi z^2} \]

Singular part: at \( z = 0 \): of the form \( \frac{c}{z^2} + \frac{c_1}{z} + \phi(z) \).

Multiply by \( z^2 \), take limit as \( z \to 0 \),

\[ c_{-2} = 1, \quad \text{and} \quad \sin^2 \pi z \text{ even, so } c_1 = 0. \]

Then by periodicity, \( \sin^2 \pi (z-n) = \sin^2 \pi z \), the singular part is the same at all integers: \( \frac{1}{(z-n)^2} \).

Thus Mittag-Leffler's Theorem tells us that

\[ \frac{\pi^2}{\sin^2 \pi z} = \sum_{n=0}^{\infty} \frac{1}{(z-n)^2} + g(z) \]

since \( \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \) is convergent for any \( z \not= n \), integer

and absolutely conv. on any compact set.

so we don't need to correct with Taylor polys.

\[ \frac{1}{\sin^2 \pi z} \to 0 \]

as \( |z| \to \infty \) uniformly.

\[ \cosh y = \frac{e^y + e^{-y}}{2} \to \infty \]

as \( |y| \to \infty \).