\[
\frac{\pi^2}{\sin^2 \pi z} = \pi^2 \csc^2 \pi z = \frac{d}{dz} \left( -\pi \cot \pi z \right)
\]

Then applying Mittag-Leffler theorem to

\[
\pi \cot \pi z = \sum_{n \in \mathbb{Z}} \left[ \frac{1}{z-n} - \phi_n(z) \right] + g(z)
\]

we draw the

Conclusion: \( \phi_n(z), g(z) \) constant.

so

\[
\phi_n(z) = -\frac{1}{n}, \text{ const. term in Taylor expansion of } \frac{1}{z-n}, \quad n \neq 0
\]

\[
\phi_0(z) = 0.
\]

Then can check directly that

\[
\sum_{n \in \mathbb{Z}} \left( \frac{1}{z-n} + \frac{1}{n} \right) = \sum_{n \neq 0} \frac{2}{n(z-n)}
\]

\[
\sum_{n \neq 0} \frac{2}{n(z-n)} = \frac{\pi^2}{\sin \pi z} - \sum_{n \neq 0} \frac{1}{z-n} + g(z) \quad \Rightarrow \quad g(z) = 0.
\]

\( g(z) = 0 \) because, grouping \( n \) and \(-n\) terms together (which is permissible since series converges absolutely, so sum is not altered by regrouping).

Example 2: \( \frac{\pi}{\sin \pi z} \). Then singular parts are

\[
\frac{(-1)^n}{(z-n)}
\]

\( (z-n) \) doesn't converge absolutely, but can prove \( \frac{\pi}{\sin \pi z} = \lim_{m \to \infty} \sum_{n=1}^{m} (-1)^m \quad \text{via cotangent identity}\).
We have addressed poles of meromorphic functions. What about zeros? Even for finitely many zeros, is there a canonical rep of all such functions?

If \( f \) is entire, never zero, then we can write

\[
f(z) = e^{g(z)} \quad g(z) \text{; entire.}
\]

\( \frac{f'}{f} \) (is analytic in whole plane, so is the derivative of entire function, call it \( g(z) \)).

\[
f(z) e^{-g(z)} \text{ has derivative identically 0.}
\]

\[
f(z) = c e^{g(z)} \text{ and can absorb constant c into } g(z).
\]

Then if \( a_1, \ldots, a_N \) : zeros (repeated according to multiplicity), then we can write:

\[
f(z) = z^m e^{g(z)} \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right)
\]


Naive guess:

If \( \exists a_i \) : possibly infinite collection of zeros of function, \( a_i \neq 0 \forall i \)

then write

\[
f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)
\]

What does it mean for infinite product to converge?

Ans: Just like for sums, consider partial products

\[
\prod_{n=1}^{N} p_n := \lim_{N \to \infty} P_n \text{ when } P_n = p_1 \cdots p_n \ldots \text{ (provided limit exists and } \neq 0)\]

(Note 0 bad because then if any \( p_n = 0 \), product converges independent of growth of factors.)

Still too restrictive: Remove the (at most finitely many) factors \( p_n = 0 \).

and ask that sequence of partial products formed from remaining \( p_n \).
if partial products $P_n$ converge, then \( \lim_{n \to \infty} \frac{P_n}{P_{n-1}} = 1 \)

and since \( \frac{P_n}{P_{n-1}} = P_n \), these \( P_n \to 1 \) as well.

Write products in form: \( \prod_{n=1}^{\infty} (1 + b_n) \) with nec. condition for convergence

\[ b_n \to 0. \]

**Key idea** regarding convergence of infinite products: Take logarithm.

Converts to infinite sum.

Then assess convergence of sum.

\[
\log \left( \prod_{n=1}^{\infty} (1 + b_n) \right) = \sum_{n=1}^{\infty} \log (1 + b_n)
\]

where we choose principal branch of logarithm here.

If \( S_N = \sum_{n=1}^{N} \log (1 + b_n) \) then \( P_N = e^{S_N} \) and hence if

\[
S_N \to S \quad \text{then} \quad P_N \to e^S \quad (\neq 0)
\]

(i.e. convergence of sum is sufficient for conv. of product)

In fact, convergence of sum of logs also necessary condition. (need to be careful about branch)

Suppose \( P_N \to P \) for. Then \( \log (P_N/P) \to 0 \) as \( N \to \infty \).

For any \( N \), \( \exists h_N \) s.t. \( \log (P_N/P) = S_N - \log P + h_N \cdot 2\pi i \)

\[
\Rightarrow (h_{N+1} - h_N) \cdot 2\pi i = \log (P_{N+1}/P) - \log (P_N/P) - \log (1 + a_N)
\]

\[
\Rightarrow h_{N+1} = h_N \quad \text{for } N \text{ suff. large, since arg of r.h.s. has }
\]

\[
(\text{call this integer } h)
\]

Conclusion: \( S_N \to \log P - h \cdot 2\pi i \)

where \( h \) as \( h = \arg (P_0/P) = S_1 - \log P + 2\pi i, \ n \gg n_0 \).

\[ |\arg (1+a_N)| \leq \pi \text{ and } \arg (P_{N+1}/P) - \arg (P_N/P) \to 0 \]
To summarize:

\[ \prod_{n=1}^{\infty} (1 + a_n) \quad (\text{where we assume } 1 + a_n \neq 0) \quad \text{converges if and only if} \]

\[ \sum_{n=1}^{\infty} \log(1 + a_n) \quad \text{(where summands represent principle branch of } \log) \]

for absolute convergence, even simpler since

\[ \sum_{n=1}^{\infty} |\log(1 + a_n)| \quad \text{converges iff} \]

\[ \sum_{n=1}^{\infty} |a_n| \quad \text{converges} \]

(either of these)

If both series converge absolutely, we say

\[ \prod_{n=1}^{\infty} (1 + a_n) \quad \text{converges absolutely} \]

Similar equivalences are true for uniform convergence on compact sets, between products and corresponding sums.

Back to our original question:

How to make sense of:

\[ f(z) = \sum_{n=1}^{\infty} e^{g(z)} \quad \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \]?

\[ |a_n| \to \infty \]

\[ |z/a_n| \to 0 \]

converges absolutely if and only if

\[ \sum_{n=1}^{\infty} \frac{|z|}{|a_n|} \quad \text{converges, i.e. if} \]

\[ \sum_{n=1}^{\infty} |a_n| \quad \text{converges} \]

(and then convergence is also uniform on compact sets.)

\[ \text{(and then convergence is also uniform on compact sets.)} \]

\[ \text{closed disks of radius } R \]

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\[ \text{f otherwise need a correction...} \]