Last week: meromorphic functions with prescribed zeros/poles (+ singular parts)

For poles: Mittag-Leffler Thm \( \exists \{a_n\} \text{ with } |a_n| \to \infty \text{ as } n \to \infty \)

\[ S_n: \text{ polynomials w/ no const. term} \]

such \( S_n \) then there exists \( f, \text{ meromorphic w/ poles } \{a_n\}, \text{ singular parts } S_n \)

Any \( f \) is of form:

\[ \sum_n \left( S_n \left( \frac{1}{z-a_n} \right) + p_n(z) \right) + g(z) \]

for some polynomials \( p_n \), and \( g \) analytic, and such that series converges absolutely (entire)

Hidden: there aren’t many good choices \( p_n \), so that series converges absolutely.

Best choice: \( p_n \): a Taylor polynomial for

most natural \( S_n \left( \frac{1}{z-a_n} \right) \text{ of sufficiently high degree} \)

in any other choice \( p_n^*(z) \) differs by abs. conv. series which can be absorbed into the \( g \).

Example: \( \pi \cot \pi z = \sum_n \left( \frac{1}{z-n} + \frac{1}{n} \right) \) (\( p_n \): const. terms of Taylor exp. of \( \frac{1}{z-n} \))

For zeros: Use products not sums. \( g = 0 \)

Fix naive guess: \( \{a_n\} \text{ with } |a_n| \to \infty \)

Any such \( f \) with zeros at \( \{a_n\} \) is of form: \( f: \text{analytic} \)

\[ f(z) = e^{g(z)} \prod_n \left( 1 - \frac{z}{a_n} \right) \]

for some \( g \) entire analytic.

Just as with sums, correct terms in product so it absolutely converges.
To summarize:

\[ \prod_{n=1}^{\infty} \left(1 + a_n\right) \] (where we assume \(1 + a_n \neq 0\)) converges if and only if

\[ \sum_{n=1}^{\infty} \log \left(1 + a_n\right) \] (where summands represent principle branch of log.)

for absolute convergence, even simpler since

\[ \sum_{n=1}^{\infty} \left| \log (1 + a_n) \right| \] converges iff

\[ \sum_{n=1}^{\infty} |a_n| \] converges.

(=think Taylor expansions.

In particular \( \lim_{z \to 0} \frac{\log (1 + z)}{z} = 1 \)

i.e. since \(|a_n| \to 0\) if (either)

scores converge absolutely, we say

\[ \prod_{n=1}^{\infty} \left(1 + a_n\right) \] converges absolutely.

Similar equivalences are true for uniform convergence on compact sets,

between products and corresponding sums.

Back to our original question:

How to make sense of:

\[ f(z) = \prod_{n=1}^{\infty} e^{g(z)} \left(1 - \frac{z}{a_n}\right) \]

converges absolutely if and only if

\[ \sum_{n=1}^{\infty} \left| \frac{z}{a_n} \right| \] converges, i.e. if \( \sum_{n=1}^{\infty} |\frac{1}{a_n}| \) converges

so \( \left| \frac{z}{a_n} \right| \to 0 \)

(\text{and thus convergence is also uniform on compact sets.})

So, \( |f(z)| \to 0 \)

\[ \text{for all } z 
\]

which gives simultaneous convergence.

Need a correction...
For poles, we had modified the singular part:

$$\sum_i P_i \left( \frac{1}{z-a_i} \right) - p_i(z)$$

by $p_i(z)$: Taylor polynomials for $P_i \left( \frac{1}{z-a_i} \right)$ at origin.

Here, given $\sum a_n$ with $|a_n| \to \infty$ as $n \to \infty$,

we ask for polynomials $p_n(z)$ such that

$$\prod_{n=1}^\infty \left( 1 - \frac{z}{a_n} \right)^{p_n(z)}$$

converges to entire function.

(Or equivalently, so that $\sum_n \left[ \log \left( 1 - \frac{z}{a_n} \right) + p_n(z) \right]$ converges.)

So take $p_n(z)$ to be the $m_n$-th Taylor poly of $\log \left( 1 - \frac{z}{a_n} \right)$ for $m_n >> 0$.

Essentially a repeat of our earlier arg.

**Thm (Weierstrass):** \( \exists \) entire function with arbitrarily prescribed zeros $\sum a_n$

provided $|a_n| \to \infty$ as $n \to \infty$ (if sequence infinite)

Every such function is of form:

$$f(z) = e^{g(z)} \prod_{n=m_1}^{m_2} \left( 1 - \frac{z}{a_n} \right)^{\frac{y}{a_n}} \cdots \frac{1}{m_n (\frac{z}{a_n})^{m_n}}$$

$m_n$ chosen so that prod. converges to entire function.

**Corollary:** If $g$ is meromorphic on $\Omega$, then

then $g = h_1/h_2$ $h_1, h_2$ analytic on $\Omega$.

**Proof:** Let $f$ as above with $a_i$ poles of $g$. Then $f \cdot g$ is entire. \( \Box \)
In fact
\[
\log\left(1 - \frac{\pi}{a_n}\right) + \phi_n(z) = -\frac{1}{m_n + 1} \left(\frac{\pi}{|a_n|}\right)^{m_n + 1} + \text{higher order terms}
\]

Fix $R$, consider only $|a_n| > R$ when analyzing convergence for $|z| \leq R$

Then \( |r_n(z)| \leq \frac{1}{m_n + 1} \left(\frac{R}{|a_n|}\right)^{m_n + 1} \left(1 - \frac{R}{|a_n|}\right)^{-1} \)

so suffices to show \( \sum_{n=1}^{\infty} \frac{1}{m_n + 1} \left(\frac{R}{|a_n|}\right)^{m_n + 1} \) converges. (**)

so \( m_n \) can be chosen so that:

(choose \( m_n = n \), for example)

Obtain geometric series. Note degree of \( \phi_n(z) \), which we've been calling \( m_n \), may vary with \( n \). But in practice, can often obtain convergence by choosing \( m_n = h \), fixed const. indep. of \( n \). (Just as in our examples with Mittag-Leffler sums.)

If so, then \( \frac{\pi}{|a_n|^{h+1}} \) may be removed from

so we require \( \sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}} \) to converge

Let \( h \) be smallest such integer (how allowed)

Then we have a canonical expression for the product:

\[
\prod_n \left(1 - \frac{\pi}{a_n}\right)e^{\frac{\pi}{a_n} + \cdots + h\left(\frac{\pi}{a_n}\right)^h}
\]

and for the function:

\[
f(z) = z^m e^{g(z)} \prod_n \left(1 - \frac{\pi}{a_n}\right)e^{\frac{\pi}{a_n} + \cdots + h\left(\frac{\pi}{a_n}\right)^h}
\]

Define genus \((f)\)

\[
\max\left(\deg(g), h\right)
\]

if \( g \) may be taken to be unimodal.
Example: \( \sin \pi z \) is zero at integers

Want smallest \( h \) such that \( \sum \frac{1}{|a_n|} z^{h+1} \) converges

i.e. \( \sum \frac{1}{n^{h+1}} \) converges. \( \Rightarrow \) Then \( h = 1 \)

\( \deg h = 1 \)

\( \downarrow \)

\( \sum \frac{1}{n^{h+1}} \)

So our canonical product takes form: \( \sin \pi z = z e^{g(z)} \prod \left( 1 - \frac{z}{n} \right) e^{\frac{z}{n}} \)

To determine genus, we need to know \( \deg(g) \).

Take logarithmic derivative on both sides (justified by convergence of product on compact sets)

\( \pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n=0} \left( \frac{1}{z-n} + \frac{1}{n} \right) \)

But from our earlier Mittag-Leffler formula, we know \( g'(z) = 0 \)

\( \Rightarrow g(z) \) constant.

Since \( \lim_{z \to 0} \frac{\sin \pi z}{z} = \pi \), then \( e^{g(0)} = \pi \)

So \( \sin \pi z = \pi z \prod \left( 1 - \frac{z}{n} \right) e^{\frac{z}{n}} \). (genus 1)

Genus telling us about growth of function.

\[ \text{order}(f) = \lim_{R \to \infty} \frac{\log \log M(R)}{\log R} \]

\[ M(R) : \text{max of } f \text{ on circle of radius } R \text{ centered at origin} \]

Then \( h \leq \text{order}(f) \leq h+1 \). \( h \): genus