Last time, showed diff. function \( f: \mathbb{C} \to \mathbb{C} \) (or \( \Omega \to \mathbb{C} \))

must satisfy Cauchy–Riemann eqns. ("2 paths argument")

\[
\frac{df}{dx} = -i \frac{df}{dy} \quad \text{(i.e. } \frac{du}{dx} = \frac{dv}{dy}, \frac{du}{dy} = -\frac{dv}{dx})
\]

If we write \( f = u + iv \)

Suppose Cauchy–Riemann eqns are true.

Is \( f \) differentiable? (Almost. Need to add one assumption)

Mid proof: Analyzing difference quotient, whose numerator included:

\[
u(x+h, y+k) - u(x,y+k) + u(x,y+k) - u(x,y)
\]

\[
\approx h \cdot \frac{du}{dx}(x,y) \quad \text{But off by error}
\]

To make precise: Use MVT

e.g. \( u(x+h, y+k) - u(x,y+k) = h \cdot \frac{du}{dx}(x+h’, y+k) \)

For some \( h’ \) in interval \((x-h, x)

So error = \( h \cdot \left[ \frac{du}{dx}(x-h’, y+k) - \frac{du}{dx}(x,y) \right]

+ k \cdot \left[ \frac{du}{dy}(x,y+k’) - \frac{du}{dy}(x,y) \right] \quad (\star)

Error from \( u’s \)

\[
\text{\( h+ik \to 0 \) as \( h+ik \to 0 \) since } \left| \frac{h}{h+ik} \right| \leq 1, \left| \frac{k}{h+ik} \right| \leq 1
\]

AND we assume that partials are continuous. (so that terms in brackets in (\( \star \)) \( \to 0 \) as \( h+ik \to 0 \).)
Similar calculation for \( v \)'s,

\[
\lim_{h \to 0} \frac{f(z + (h + it)k) - f(z)}{h + itk} = \left[ h \cdot \frac{\partial u}{\partial x}(x, y) + k \frac{\partial u}{\partial y}(x, y) \right] + i \left[ h \cdot \frac{\partial v}{\partial x}(x, y) + k \frac{\partial v}{\partial y}(x, y) \right]
\]

Now use C.R to make sense of this limit:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

So get upon substituting:

\[
\lim_{h \to 0} \frac{h(k \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x})}{h + itk}
\]

whose limit is now clear!

Cor: \( f \) diff., then \( \Delta u = \Delta v = 0 \).

(of easy direction) \( f = u + iv \)

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right)
\]

\[
\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left( -\frac{\partial v}{\partial x} \right)
\]

Mixed partials are equal if \( v \) has continuous first partials so these cancel.

So \( u, v \) are "harmonic."

Could give alternate equivalence:

\( f \) diff \( \iff \) \( u, v \) harmonic + satisfy C-R equations

Very basic PDE with interesting applications, see for example Dirichlet problem in Ch.6 of Ahlfors.
Polynomials: Stated previously that polyc. in $z=x+iy$ are differentiable since $f(z) = c$, $c$ = constant, $f(z) = z$ are diff. and we have sum, product rules.

Given location of roots of $f(z)$, what can be said about the roots of $f'(z)$? For functions $\mathbb{R} \to \mathbb{R}$, question doesn't have a good answer. (E.g. consider shifting parabola like $f(x) = x^2 - 1$ upward/downward may or may not have real roots, but derivative always has a root at $x = 0$)

Assumption: Fundamental Thm. of Algebra
(to be proved on p. 122 of Ahlfors upon developing complex integration theory)

Thus any polynomial $P(z) = a_n z^n + \cdots + a_1 z + a_0$

can be expressed in form $P(z) = a_n (z - d_1) \cdots (z - d_m)$

$\text{dn} : \text{roots (or "zeros")}$

Thm (Lucas-Gauss): If zeros of $P(z)$

are contained in polygon in $\mathbb{C}$, then roots of $P'(z)$ are (zeros)

contained in some polygon.
Pr: Sufficient to show that if zeros of \( P(z) \) lie in half plane \( H \),
then zeros of \( P'(z) \) lie in \( H \) as well. I.e. \( z \notin H \Rightarrow P'(z) \neq 0 \)
( After all, any polygon is just finite intersection of half planes )

Key idea: Show if \( z \notin H \), \( \frac{P'(z)}{P(z)} \neq 0 \).

Note \( \frac{P'(z)}{P(z)} = \frac{1}{z-d_1} + \cdots + \frac{1}{z-d_n} \) di roots.

d (direct consequence of product rule. e.g. \( n=2 \) , \( P(z) = (z-d_1)(z-d_2) \)
\( P'(z) = (z-d_1)' + (z-d_2)' \).

Easy example: \( H \): lower half plane \( \{ z \mid \text{Im}(z) < 0 \} \)

\[ z \in \text{Im}(z) < 0 \]

\[ z \notin \text{Im}(z) < 0 \]

\[ \text{Im}(z-d_i) \]

\[ \text{Im}(\frac{1}{z-d_i}) \]

positive, so reciprocal \( \text{Im}(\frac{1}{z-d_i}) \) negative

so \( \frac{P'(z)}{P(z)} \) has negative imaginary part.

General case is same idea.

Now line in \( \mathbb{C} \) has parametric equation \( z = a + bt \), \( a, b \in \mathbb{C} \) fixed
the \( \mathbb{R} \) parameter

dividing \( \mathbb{C} \) into half planes \( \text{Im}(\frac{z-a}{b}) < 0 \) call this \( H \).
\[ \text{Im}(\frac{z-a}{b}) > 0 \]

Note \( \text{Im}(\frac{z-d_k}{b}) = \text{Im}(\frac{z-a}{b}) - \text{Im}(\frac{a-d_k}{b}) \).

\( f \ z \notin H \), then \( >0 \) and \( <0 \) so \( \text{Im}(\frac{z-d_k}{b}) > 0 \)
Hence \( \text{Im} \left( \frac{b}{z - d_k} \right) < 0 \) so \( \frac{b P'(z)}{P(z)} = \sum_{k=1}^{n} \frac{b}{z - d_k} \) has negative imaginary part.

so \( P'(z) \neq 0 \). \( \checkmark \)

Rational functions: Again differentiable if \( \sum Q(z) \neq 0 \) in \( \frac{P(z)}{Q(z)} \).

Derivative given by quotient rule.

Places where \( Q(z) = 0 \) are called "poles" of rational function \( \frac{P(z)}{Q(z)} \).

If we consider \( R(z) = \frac{P(z)}{Q(z)} : \mathbb{C} \cup \{0\} \to \mathbb{C} \cup \{0\} \)

(can check it is continuous at \( \infty \) using our metric from stereographic proj.)

Set \( R(z) = \lim_{z \to \infty} R(z), \) but this doesn't allow one to determine order of zero or pole.

Better: Change of coordinates \( z \mapsto \frac{1}{z} \)

\( \infty \mapsto 0 \)

Analyze function \( R \left( \frac{1}{z} \right) \) at \( z = 0 \).

Example: \( R(z) = \frac{z^2 - 1}{z^3} \) has zeros at \( 1, -1 \) (order 1)

pole of order 3 at \( z = 0 \)

If \( z = \infty \), analyze \( R \left( \frac{1}{z} \right) = \frac{\frac{1}{z^2} - 1}{\frac{1}{z^3}} = z - z^3 \) so zero (of order 1) at \( \infty \).

*: Note that we always want to consider \( R(z) = \frac{P(z)}{Q(z)} \) in reduced form so that \( P, Q \) have no common factors.
Proposition: As a function on $\mathbb{C} \cup \{\infty\}$, $R(z)$ has equal number of zeros, poles (counted with multiplicity) and this equals $\max(\deg P, \deg R)$.

Proof: Bookkeeping at $z = \infty$ + Fundamental Theorem of Algebra.

More interesting way to form analytic (i.e. holomorphic, differentiable) functions.

Use Power Series.

(Read proof in Ahlfors 2.2.1 on completeness of $\mathbb{C}$.)

sequence converges iff it is Cauchy.

Just as with $\mathbb{R}$, series converge if $\lim$ of partial sums converges.

There are stronger notions of convergence:

Say $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

(If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges by $\Delta$-inequality.)

Nice fact about absolutely convergent series: Can rearrange order of terms without affecting the sum of the series (not so in general).

(Asked to prove this for next week's problem set.)