Last time: \( f \text{ diff } \iff u,v \text{ satisfying C-R eqns } + \text{ first partials continuous} \)

Two examples: polynomials, rad'll functions.

generalize them: power series, Laurent series (allowed to have finitely many terms in negative powers of \((z - z_0)\))

and show that holomorphic functions \(\uparrow\) meromorphic functions \(\downarrow\)

(to be defined later by generalizing def. of pole)

Today, want to prove that power series define holomorphic functions, and derivatives given by term-by-term diff. in sense.

Big Plan for Course: This week - finish ex. functions

Wed: Exp/Log, Fri: Finish Exp/Log, discuss topology very briefly §3.1 in Ahlfors

Next week: Start integration theory (skip Moebius transformations, confirmed maps for now.)
Proposition: As a function on \( \mathbb{C} \cup \mathbb{R}^3 \), \( R(z) \) has equal number of zeros, poles (counted with multiplicity) and this equals \( \max(\deg P, \deg R) \).

**pf:** Bookkeeping at \( z = \infty \) - Fundamental theorem of algebra.

More interesting way to form analytic (i.e. holomorphic, differentiable)

Use Power Series.

(Read proof in Ahlfors 2.2.1 on completeness of \( \mathbb{C} \):)

sequence converges iff it is Cauchy

Just as with \( \mathbb{R} \), series converge if \( \sum \) of partial sums converges.

There are stronger notions of convergence:

Say \( \sum a_n \) converges absolutely if \( \sum |a_n| \) converges.

(If \( \sum a_n \) converges absolutely, then it converges by \( \Delta \)-inequality)

Nice fact about absolutely convergent series: Can rearrange order of terms without affecting the sum of the series (not so in general)

(Asked to prove this for next week's problem set)
Uniform convergence: Given sequence of functions $\xi f_n(x)$ for all $x \in E$, some set, if $f_n(x) \to f(x)$ for some $x$,
then given $\epsilon > 0$, $\exists N := N(x)$ s.t.

$$|f_n(x) - f(x)| < \epsilon \quad \text{if} \quad n \geq N.$$ 

If $N$ may be chosen independent of $x \in E$, then we say $\xi f_n(x)$ converges uniformly to $f(x)$.

Nice non-example from Ahlfors: $\lim_{n \to \infty} (1 + \frac{1}{n})x = x$

where $N > \frac{|x|}{\epsilon}$.

Important property of uniformly convergent sequences:

if $\xi f_n(x)$ converges uniformly, $f_n(x)$ continuous for all $n$,
then the limit is continuous.

pf: Suppose $f_n(x)$ continuous, $\xi f_n$ converges uniformly to $f$.
For any $\epsilon > 0$, we can find $n$ s.t.

$$|f_n(x) - f(x)| < \frac{\epsilon}{3} \quad \text{for all} \quad x \in E$$

Since $f_n$ continuous at $x_0$, $\exists \delta > 0$ s.t.

$$|f_n(x) - f_n(x_0)| < \frac{\epsilon}{3} \quad \text{if} \quad |x - x_0| < \delta.$$ 

Thus if $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \epsilon.$$
Natural test for uniform convergence (for series)

Weierstrass M-test: Let \( \sum a_n \) be convergent series \((a_n \geq 0)\)

If \( |f_n(x)| \leq M \cdot a_n \), \( M \): constant

for all sufficiently large \( n \),

(for all \( x \in E \): set) then \( f_n \) converges uniformly on \( E \).

If: comparison test.

Power series: generally of form \( f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \), \( a_n \in \mathbb{C} \)

(*: see note)

"expansion at \( z_0 \)"

Classic example: geometric series \( \sum_{n=0}^{\infty} z^n \) since we have closed form for partial sums: \( S_k = 1 + \ldots + z^k \)

so converges for \( |z| < 1 \), diverges \( = \frac{z^{k+1} - 1}{z - 1} \)

for \( |z| > 1 \).

Ranks: 
1. Reason for divergence different for \( |z| = 1 \) vs. \( |z| > 1 \).
2. Region of convergence (a disc) is typical, but having closed form for solution is not.

*: In practice, we give proofs only for "expansions at \( z_0 = 0 \)" since makes notation a bit less messy. Of course, all statements remain...
In fact, most convergence tests for general power series are obtained by comparison with geometric series.

Idea: compare \( |a_n z^n| \) with \( x^n \) (for \( x < 1 \), positive) if we can show that, for \( n \) suff. large,

\[
|a_n z^n| < x^n < 1, \quad \text{then } \sum_{n=0}^{\infty} a_n z^n \text{ converges absolutely.}
\]

\[
\sqrt[n]{|a_n|} \cdot |z| < x < 1 \text{ for } n \text{ suff. large.}
\]

Find \( z \) for which this holds. Naive guess: \( |z| = \lim_{n \to \infty} \sqrt[n]{|a_n|} \)

But \( \lim_{n \to \infty} \sqrt[n]{|a_n|} \) may not exist. Better to use \( \limsup \) instead, "\( \limsup \) for short.

\[
\lim \sup_{n \to \infty} x_n = \inf_{n \to \infty} \sup_{k \geq n} x_k := l \in [-1, 1]
\]

Better to understand defining property: ① For any \( \varepsilon > 0 \), \( \exists N \text{ s.t.} \)

\[
x_n < l + \varepsilon \quad \forall n > N.
\]

⇒ (This is what we need for pt. convergence/div.)

② For any \( \varepsilon > 0 \), \( \exists N \text{ s.t.} \)

\[
x_n > l - \varepsilon \quad (\text{i.e. arbitrarily large } n \text{ for which } x_n > l - \varepsilon).
\]

(\( \text{switch roles of } \inf / \sup \text{ in definition of } \lim \sup \text{ to get } \lim \inf \))
So let's start over... Let \( R = \lim_{n \to \infty} \frac{1}{\sqrt[n]{|a_n|}} \).

**Theorem 1:** \( \sum_{n=0}^{\infty} a_n z^n \) converges absolutely for \( |z| < R \)

and uniformly for \( |z| \leq p < R \).

(2) The series diverges for \( |z| > R \) (because terms grow unbounded)

**pf of 1:** If \( |z| < R \), then \( \exists x \) with \( |z| < x < R \)

so \( \frac{1}{x} > \frac{1}{R} \) \( \Rightarrow \) \( \exists N \) s.t. \( |a_n|^{\frac{1}{n}} < \frac{1}{x} \) \( \forall n > N \)

i.e. \( |a_n| < \frac{1}{x^n} \) \( \forall n > N \).

so \( |a_n z^n| < \left( \frac{|z|}{x} \right)^n \) for \( n > N \) so power series converges by comparison test.

For uniform convergence, if \( |z| \leq p < R \),

then pick \( x \) s.t. \( p < x < R \). Then \( |a_n z^n| < \left( \frac{p}{x} \right)^n \)

so convergence is uniform by Weierstrass M-test.

**pf of 2** if \( |z| > R \) choose \( x \) with

\( R < x < |z| \), so \( \frac{1}{x} < \frac{1}{R} \). By other defining prop for \( \lim \sup \), there are only many \( n \) s.t. \( |a_n|^{\frac{1}{n}} > \frac{1}{x} \).
\[ \iff [a_n] > \frac{1}{x^n} \quad \text{so} \quad |a_n z^n| > (\frac{121}{x})^n \quad \text{for infinitely many } n \] 

Terms of series grow without bound.

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**Theorem, Part 3**: For \( |z| < R \), \( \sum_{n=0}^{\infty} a_n z^n = f(z) \) defines an analytic function, with derivative given by term-by-term differentiability.

The same radius of convergence for \( \sum_{n=1}^{\infty} n a_n z^{n-1} \) as for \( f \).

**If**: Note that \( \sqrt[n]{n a_n} \) has same \( \limsup \) as \( \sqrt[n]{a_n} \)

since \( \lim_{n \to \infty} \sqrt[n]{n} = 1 \). (see Ahlfors for cute pf. using binomial thm., p. 39)

For \( |z| < R \), write

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n = s_k(z) + R_k(z) \]

Then

\[ \sum_{n=1}^{\infty} n a_n z^{n-1} = \lim_{k \to \infty} s'_k(z) \]

since this is just derivative of finite polynomial.

Want to show \( f'(z) = d(z) \). Back to the definition of differentiability...
\[
\frac{f(z) - f(z_0)}{z - z_0} - d(z_0) = \left[ \frac{S_k(z) - S_k(z_0)}{z - z_0} - S_k'(z_0) \right] \quad (A)
\]

\[
+ S_k'(z_0) - d(z_0) \quad (B) \quad \text{(identity true for any } k) \\
+ \frac{R_k(z) - R_k(z_0)}{z - z_0} \quad (C)
\]

Strategy: Show each of three pieces are small. (for } k \text{ sufficiently large) as } z \to z_0

\(A\): definition of derivative guarantees \( for \ any \ \epsilon \)

\[\exists \ \delta \ \text{ s.t. } |z - z_0| < \delta \implies A < \frac{\epsilon}{3}.\]

\(B\): \[\lim_{k \to \infty} S_k'(z_0) = d(z_0) \quad \text{so } \exists \ N_B \ \text{s.t. } k > N_B \]

\[\text{then } B < \frac{\epsilon}{3}.\]

\(C\): \[\left| \frac{R_k(z) - R_k(z_0)}{z - z_0} \right| \leq \sum_{n=k}^{\infty} |a_n| \left| \frac{z - z_0}{z - z_0} \right|^{n-k+1} \]

\[\leq \frac{|z - z_0|^{1}}{z - z_0} + \frac{|z - z_0|^{2}}{(z - z_0)^2} + \ldots + \frac{|z - z_0|^{k-1}}{(z - z_0)^{k-1}}\]

\[< \frac{|z - z_0|^{1}}{z - z_0} + \frac{|z - z_0|^{2}}{(z - z_0)^2} + \ldots + \frac{|z - z_0|^{k-1}}{(z - z_0)^{k-1}}\]

if \[|z - z_0| < R \] pick \(x\)

\[
s.t. \quad |z - z_0| < x < R
\]

so \(C \leq \sum_{n=k}^{\infty} |a_n| n \cdot x^{n-1} < \infty \). So pick \(N_c\)

\[\text{so } C < \frac{\epsilon}{3}.\]
so choose \( k \geq \max(N_B, N_c) \), and we have shown
\[
\left| \frac{f(z) - f(z_0)}{z - z_0} - d(z_0) \right| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta.
\]
so derivative exists and is equal to \( d(z_0) \) for \( |z_0| < R \).

corollary: \[ a_k = \frac{f^{(k)}(z_0)}{k!} \quad \text{(Taylor's formula)} \]
Exploring examples of differentiable functions:

Prove differentiability of nice classes of functions directly from the definition—e.g., polynomials, rational functions, power series.

For power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) has radius of convergence \( R \) with \( \frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n} \) s.t. for \( |z| < R \), \( f(z) \) converges absolutely, differentiable, with derivative given by term-by-term differentiation:

\[
 f'(z) = \sum_{n=1}^{\infty} n \cdot a_n z^{n-1} \quad \text{(with same radius of conv. } R \text{)}
\]

Can REPEAT this process, proving power series are inf. diff. for \( |z| < R \) with \( f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k} \)

\[
 = k! \cdot a_k + \ldots
\]

so \( f^{(k)}(0) = k! \cdot a_k \iff a_k = \frac{f^{(k)}(0)}{k!} \) and hence

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.
\]

CAUTION: haven't proved anything about arbitrary analytic functions, only those initially defined cs a power series.