Exploring examples of differentiable functions:

Prove differentiability of nice classes of functions directly from the definition—e.g., polynomials, rational functions, power series.

For power series \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) has radius of convergence \( R \)

\[
\frac{1}{R} = \limsup_{n} |a_n|^{1/n}
\]

s.t. for \( |z| < R \), \( f(z) \)
converges absolutely, differentiable, with derivative given by term-by-term differentiation:

\[
f'(z) = \sum_{n=1}^{\infty} n \cdot a_n z^{n-1}
\]

(\text{with same radius of conv. } R)

Can REPEAT this process, proving power series are inf. diff. for

\( |z| < R \)

with \( f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k} \)

\[
= k! \cdot a_k + \ldots
\]

so \( f^{(k)}(0) = k! \cdot a_k \leftrightarrow a_k = \frac{f^{(k)}(0)}{k!} \) and hence

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n
\]

CAUTION: Haven't proved anything about arbitrary analytic functions, only those initially defined as a power series.
Where do (important examples of analytic functions via) power series come from?

**Answer:** Solutions to linear ODEs. E.g. of the form

$$a_n(z) f^{(n)}(z) + \cdots + a_1(z) f'(z) + a_0(z) f(z) = 0$$

(let's assume $a_n(z) \neq 0$ for simplicity) with or without initial conditions.

Substitute using $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and solve via recursive relations for $a_n$'s.

**Example** $f(z) = f'(z)$ with initial condition $f(0) = 1$

Then $a_0 + a_1 z + a_2 z^2 + \cdots = a_1 + 2a_2 z + 3a_3 z^2 + \cdots$

$$\Rightarrow \quad a_{n-1} = n \cdot a_n \quad \Rightarrow \quad a_n = \frac{1}{n!}$$

$a_0 = a_1 = 1$ (from initial cond.)

So solution: $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$ (owing to its agreement with $e^x$ for $x$ real.)

**Questions:**

1. Where does it converge? What is $R$?

2. What are its properties? Slightly trolly since defined as infinite series.

(not initially defining $e$ as limit, or using inverse of logs, etc.)
for convergence, we compute $R$ via

$$\limsup_{n \to \infty} \left| \frac{1}{n!} \right|^\frac{1}{n} = 0 \quad \text{so} \quad R = \frac{1}{0} = \infty$$

(we are using root test to determine convergence. Can also use ratio test, proving that resulting $R$ is same for root test since we already established all properties of power series from root test)

prove limit by crude estimate for $n!$ e.g. show $n! > \left( \frac{n}{4} \right)^n$, for $n$ suff. large.

In any case $R = \infty$, so $e^z$ converges for all $z \in \mathbb{C}$.

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Properties: 1. Additivity $e^a e^b = e^{a+b} \quad \forall \ a, b \in \mathbb{C}$

pf 1: Use thm. of Cauchy on multiplication of abs. convergent power series (Whittaker-Watson 2-53)

$$(1 + a + a^2/2 + \cdots)(1 + b + b^2/2 + \cdots) = 1 + (a+b) + \left( \frac{a^2}{2} + ab + \frac{b^2}{2} \right) + \cdots$$

pf 2: (prettier) Use differential equation + product rule. Given any $c \in \mathbb{C}$

$$D(e^z, e^{c-z}) = e^z e^{c-z} + e^z (-e^{c-z}) = 0 \quad \forall \ z \in \mathbb{C}$$

claim: if $f'(z) = 0 \ \forall \ z \in \mathbb{C}$, then $f(z)$ constant.
pf of claim: \( f = u + iv \) approach along real, imag partus.

\[
\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0, \quad -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}
\]

all identically 0.

Using thm. from real-var. calculus, \( u, v \) constant on every horizontal, vertical line, hence constant on \( C \).

so \( e^2 e^{-z} = \text{constant} \). Setting \( z = 0 \), this is \( e^c \).

Let \( z = a \), \( c = a + b \) so result follows.

Corollaries:

\( I \) \( e^z e^{-z} = 1 \Rightarrow e^z \text{ never } 0 \).

\( II \) \( e^z \) has real coeffs in power series \( \Rightarrow e^z = e^{\alpha z} \)

so \( |e^z|^2 = e^z e^z = e^{2z} \). If \( z = iy \), then

\( \Rightarrow |e^{iy}|^2 = 1 \), that is, \( |e^{iy}| = 1 \) \( \Rightarrow \) Try to understand \( e^{iy} \) more precisely.

\( e^z \) and trigonometric functions:

Define \( \sin z = \frac{e^{iz} - e^{-iz}}{2i} \), \( \cos z = \frac{e^{iz} + e^{-iz}}{2} \)

Motivation: Power series for sine, cosine match their real counterparts.

E.g. \( \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \)
A little algebra from these definitions shows that:

1. \( e^{iz} = \cos z + i\sin z \), \( \forall z \in \mathbb{C} \)

In particular: \( e^{iy} = \cos y + i\sin y \) if \( y \in \mathbb{R} \).

(By uniqueness of power series rep’n, \( \cos y \) & \( \sin y \) are our familiar functions of real variable with geometrical interpretation.

Important that their definitions as \( c \times \) functions make no use of geometry.)

2. \( \cos^2 z + \sin^2 z = 1 \) \( \forall z \in \mathbb{C} \)

3. \[
\begin{align*}
D(\sin z) &= \cos z \\
D(\cos z) &= -\sin z
\end{align*}
\]

(term-by-term diff. of power series)

4. Other trig functions are thus rational functions in \( e^{iz} \).

\[
\tan z = \frac{\cos z}{\sin z} = -i \left( \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} \right)
\]

5. Additivity of \( e^{z} \) gives addition formulas for sine/cosine

\[
e.g. \ \sin (a+b) = \cos a \sin b + \sin a \cos b.
\]
Final aside: Try other examples of linear ODEs:

Legendre Equation: 
\[(1 - z^2) f''(z) + 2z f'(z) + \alpha (\alpha + 1) f(z) = 0\]

If \(\alpha\) non-neg. integer, 
result is Legendre polynomials.

(ODE is natural because it arises from studying Laplace equation 
in spherical coordinates)

Any benefit to studying Legendre polynomials as functions of a complex variable? As function of real variable, interpretation as orthogonal polynomials.
Picking up on differential equations perspective, could also define $\sin/cos : f''(z) + f(z) = 0$

as solutions to

( has a two dimensional space of solutions spanned by $e^{iz}, e^{-iz}$
which either follows from power series method of substitution or just by noting $e^{z}$ solves $f''(z) = f(z)$ plus chain rule )

To define two basis vectors for this space of solutions, might ask for power series with real coefficients (or even/odd symmetry)

Similarly, solutions to $f''(z) = f(z)$ result in hyperbolic sine/cosine.

e.g. $\cosh z = \frac{e^z + e^{-z}}{2}$ (even sol'n).

Two important topics remaining on exponential function:  
1. periodicity
2. inverse function

Definition: A function $f(z)$ is said to be periodic with period $c$ if $f(z+c) = f(z) \ \forall \ z \in \mathbb{C}$ ($c \neq 0$)

In pictures,

```
\begin{center}
\begin{tikzpicture}
\draw[->] (-3,0) -- (3,0);
\draw[->] (0,-3) -- (0,3);
\filldraw[color=red, fill=red!10, thick] (0,0) circle (2pt);
\end{tikzpicture}
\end{center}
```

just need to understand $f$

on this strip (or any strip parallel to it of width $|c|$)

"fundamental domain" for $f(z)$

( Even more interesting: two periods $c_1, c_2$
linearly indep. over $\mathbb{R}$.) Here fundamental domain is lattice in $\mathbb{R}^2 \leftrightarrow \mathbb{C}$
Notice that if \( f(z + c) = f(z) \), then 
\[ f(z + 2c) = f(z), \quad \forall z \in \mathbb{C} \]
and so \( f(z) = f(z + ck), \quad k \in \mathbb{Z}. \)

Let's show \( e^z \) has a period. If \( c \) is a period for \( e^z \), then by definition 
\[ e^{z+c} = e^z \quad \forall z \in \mathbb{C} \quad \Rightarrow \quad e^c = 1. \]

We know that \( |e^z| = e^x \) if \( z = x + iy \), so \( c \) must be pure imaginary.

Hence we must find \( y \in \mathbb{R} \) s.t. \( e^{iy} = 1. \)

(We know \( e^{iy} = \cos y + i \sin y \), so want \( y \) such that \( \cos y = 1 \) and \( \sin y = 0 \).

Geometrical intuition says \( y = 2\pi k, \quad k \in \mathbb{Z} \).

Not enough for Ahlfors. What is \( \pi \)? Prove this analytically.

**Step 1:** Show \( \exists \ c \) such that \( e^c = 1. \)

First, we have basic estimate that

\[ \sin y < y \quad \text{if} \quad y > 0 \]

(since \( \sin y = y \) at \( y = 0 \) and \( D(\sin y) = \cos y \leq 1 \))

so can prove \( e^c = 1 \) using integration.

Similarly \( D(\cos y) = -\sin y > -y \)

by estimate above

and \( \cos 0 = 1 \quad \Rightarrow \quad \cos y > 1 - \frac{y^2}{2} \quad \text{(via integration)}\)
Using \( \cos y > 1 - y^2/2 \), then integrating \( \int_0^y \cos t \, dt \), \( \int_0^y (1 - e^{-t^2}/2) \, dt \), for \( y > 0 \),

we get \( \sin y > y - y^3/6 \Rightarrow \cos y < 1 - y^2/2 + y^4/24 \).

Setting \( y = \sqrt{3} \), we find \( \cos \frac{\sqrt{3}}{2} < -1/8 \), while \( \cos 0 = 1 \).

\( \Rightarrow \exists \ y_0 \) such that \( \cos y_0 = 0 \).

( in \( (0, \sqrt{3}) \))

Since \( \cos^2 y_0 + \sin^2 y_0 = 1 \Rightarrow \sin y_0 = \pm 1 \Rightarrow e^{iy_0} = \pm i \)

\( \Rightarrow \ e^{4iy_0} = 1 \). Conclusion: \( (4y_0)i \) is a period!

**Step 2**: Show \( 4y_0i \) is smallest period. (Know pure imag, so we mean smaller than \( 4y_0 \).

Suppose there were smaller, write it as \( 4y \) for some \( y \in (0, y_0) \).

Then \( \sin y > 0 \) (since \( \sin y > y - y^3/6 = y \cdot (1 - y^2/6) > y/2 > 0 \))

(\( \text{strictly} \))

\( \Rightarrow \) cos decreasing on \( (0, y_0) \)

\( \Rightarrow \) sin increasing on \( (0, y_0) \) (since \( \sin^2 x + \cos^2 x = 1 \))

\( \Rightarrow \) \( \sin y < \sin y_0 = 1 \) so \( 0 < \sin y < 1 \).

\( \Rightarrow \) \( e^{iy} \neq \pm 1, \pm i \Rightarrow e^{i(4y)} \neq 1 \) (contradiction)
Finally we arrive at the definition of $\pi$: $2\pi := 4y_0$. (i.e. determined in terms of smallest period)

Along the way, we showed $e^{i\pi/2} = i$.

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Step 3: Show all periods are integer multis. of $2\pi$.

If $\omega$ is another period, then we can find $k \in \mathbb{Z}$ s.t.

$$2\pi k \leq \omega < 2\pi (k+1).$$

If $\omega \neq 2\pi k$, then

$$(\omega - 2\pi k)i$$

is another positive period, contradicting the minimality of $2\pi$.

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Inverse functions: Try to define a function $z = \log w$ according to $w = e^z$.

Problems: ① $e^z \neq 0 \forall z \in \mathbb{C}$, so $\log (0)$ not defined.

② If $w \neq 0$, then $|w| = |e^z| = e^x$

so $e^{iy} = w/|w|$. The equation $e^x = |w|$ has a unique solution $x = \log |w|$ (here: real logarithm).

But $e^{iy} = w/|w|$ on complex unit circle has infinitely many solutions (differing by multiples of $2\pi$).