More examples of Riemann surfaces:

(conn., Hausdorff)

Just as $\mathbb{P}^1(\mathbb{C})$ is one-dim'l cx. manifold (a.k.a. Riemann surface)

similarly $\mathbb{P}^n(\mathbb{C})$ is n-dim'l cx. manifold.

Here $\mathbb{P}^n(\mathbb{C}) \subseteq \mathbb{C}^{n+1} \setminus \{0\} / \sim$ where $[x_1, \dots, x_{n+1}] \sim [\lambda x_1, \dots, \lambda x_{n+1}]$ where $\lambda \in \mathbb{C}^\times$

with $[x_1 : \dots : x_{n+1}]$ denoting an equivalence class

charts:

$\phi_1 : U_1 \longrightarrow \mathbb{C}^n$

$[x_1 : \dots : x_{n+1}] \longmapsto (x_2/x_1, \dots, x_{n+1}/x_1)$, etc.

with $x_1 \neq 0$

$[1 : a_1 : \dots : a_n] \longleftarrow (a_1, \dots, a_n)$

Compact since $\mathbb{P}^n(\mathbb{C})$ covered by closed unit disks in U_i.

(i.e. union of finitely many compact sets)

Want to make Riemann surfaces from projective space. Cut out by $n-1$ equations.

e.g. $\mathbb{P}^2(\mathbb{C})$, with one equation. $F(x,y,z)$ is homogeneous poly. of 3 cx. variables

(homog.: degree of all monomials are the same)

F homogeneous $\Rightarrow$

$F(x,y,z) = F(\lambda x, \lambda y, \lambda z) \cdot \lambda^d$

where d is degree of all monomials appearing

so F is not constant on equivalence classes in $\mathbb{P}^2(\mathbb{C})$

but non-zero-ness of F is preserved on equivalence class

So well-defined to set $X = \{ [x:y:z] \in \mathbb{P}^2(\mathbb{C}) \mid F(x,y,z) = 0 \}$

Let $X_1 = X \cap \mathcal{U}_1$ (so $x \neq 0$, choose representative $[1:y:z]$, $y,z \in \mathbb{C}$

so seek solutions $F(1,y,z) = 0$

this is polynomial in two variables

so have affine plane curve)

X: projective plane curve.

X is "non-singular" or "smooth" if no common solutions to

$$F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$ (i.e. pts. on X where all partials vanish)

Show X is ~~projective~~ Riemann surface by showing X_i have compatible complex structures.

Lemma: F: homogeneous of deg. d. Then X smooth $\iff$ X_i smooth for all i

pf: Suppose X_1 not smooth. Then $\exists$ solution (y_0, z_0) to

$$F([1:y:z]) = \frac{\partial F}{\partial y}([1:y:z]) = \frac{\partial F}{\partial z}([1:y:z]) = 0.$$

But then $$\frac{\partial F}{\partial x}([1:y_0:z_0]) \overset{(*)}{=} \left(d \cdot F - y_0 \cdot \frac{\partial F}{\partial y} - z_0 \frac{\partial F}{\partial z}\right)[1:y_0:z_0] = 0$$

since F homog.

So X not smooth. Other direction even easier.

(*) is called "Euler's formula": $$F = \frac{1}{d} \sum_i x_i \frac{\partial F}{\partial x_i}$$

So X_i smooth $\Rightarrow$ X_i Riemann surfaces. On to compatibility:

Suppose $p \in X_1 \cap X_2 \Rightarrow x, y \neq 0$. charts for affine curves were given by projection.

ϕ_1 on X_1 could be y/x or z/x, ϕ_2 on X_2: x/y or z/y.

Now we need to do cases, analyzing $\phi_2 \circ \phi_1^{-1}$, show it is holomorphic at p.

e.g. $\phi_1([1 : y/x : z/x]) = z/x$ then $\phi_1^{-1}(w) = [1 : g(w) : w]$

$\phi_2([x/y : 1 : z/y]) = x/y$

$w \in \mathbb{C}$

according to earlier investigation.

Since locally a graph near w.

g: holomorphic.

$$\phi_2 \circ \phi_1^{-1}(w) = \phi_2([1 : g(w) : w])$$

$$= \phi_2([1/g(w) : 1 : w/g(w)])$$

$$= 1/g(w)$$ which is holom since $g(w) \neq 0$ if $[1 : g(w) : w]$ is in X_2.

Thus $X \subseteq \mathbb{P}^2(\mathbb{C})$ is Riemann surface (compact since $\mathbb{P}^2$ compact, X closed)

What about $\mathbb{P}^n(\mathbb{C})$ and ~~n~~ $n-1$ equations?

Again use homogeneous polynomials, again need non-singular condition:

$(n-1) \times (n+1)$ matrix of partial derivs $\partial F_i / \partial x_j$ to have maximal rank $(n-1)$ at each point $p \in X$.

result is called "smooth complete intersection curve" ↖ intersection of $n-1$ hypersurfaces.

Of course, being Riemann surface is local condition, so only locally needs to be described by $n-1$ hypersurfaces with non-singular condition.

Clever example from Miranda ("twisted cubic in $\mathbb{P}^3$")

Image of $[x : y] : \mathbb{P}^1$ in $\mathbb{P}^3$ given by $[x^3 : x^2 y : x y^2 : y^3]$

Defining equations for $[x_0 : x_1 : x_2 : x_3]$ are

$$x_0 x_3 = x_1 x_2$$
$$x_0 x_2 = x_1^2$$
$$x_1 x_3 = x_2^2$$

Independent equations unless we know that one of $x_i \neq 0$. E.g. $x_0 \neq 0$ then third follows from first two.