Final lemma: if V: vector space with countable basis $\{e_1, e_2, \dots\}$

then every linearly independent set of vectors is at most countably infinite.

pf: If L: linearly indep. subset of V, $V_n = \langle e_1, \dots, e_n \rangle$

Let $L_n = L \cap V_n$. This is linearly indep., $\subseteq V_n$, hence finite.

But $L = \bigcup_n L_n$ so at most countably infinite. //

START OF MONDAY LECTURE

Explore consequences of nullstellensatz:

V: variety defined as zero locus of $f_1(\underline{x}), \dots, f_r(\underline{x})$

I: ideal generated by $\langle f_1, \dots, f_r \rangle$. Then there exists a bijection

$$\left\{ \begin{matrix} \text{max. ideals} \\ \text{of } \mathbb{C}[\underline{x}]/I \end{matrix} \right\} \underset{\text{bij.}}{\longleftrightarrow} \left\{ \begin{matrix} \text{pts of} \\ V \end{matrix} \right\}$$

pf: maximal ideals of $\mathbb{C}[\underline{x}]/I \longleftrightarrow$ maximal ideals of $\mathbb{C}[\underline{x}]$ containing I (image under projection to I)

An ideal contains $I \iff$ it contains $f_1, \dots, f_r$.

and maximal ideals (by Nullstellensatz) are $\ker(\phi_{\underline{a}})$ for $\underline{a} \in \mathbb{C}^n$.

So just need to determine condition for $f_i \in M_{\underline{a}}$. ($\iff f_i(\underline{a}) = 0$ i.e. $\iff \underline{a} \in V$)

Classical form of Nullstellensatz: V = variety of zeros of $f_1, \dots, f_r$

I: ideal $\langle f_1, \dots, f_r \rangle$. If $g \equiv 0$ on V then

$g^m \in I$ for some m

Need one additional fact: Every ideal I of ring R with I not unit ideal is contained in a maximal ideal.

(follows from Zorn's lemma)

inductive partially ordered set has maximal elt

every totally ordered subset has upper bound

or can use Hilbert basis theorem since we're discussing polynomial rings. see (*) at end of notes

Then we can prove following corollary of Nullstellensatz:

Cor: $f_1, \dots, f_r$ polynomials in $\mathbb{C}[\underline{x}]$ If $f_1 = \dots = f_r = 0$ has no solution in $\mathbb{C}^n$, then $\exists\, g_i$, $i = 1, \dots, r$ with

$$1 = \sum_i g_i f_i$$

pf: Nullstellensatz $\Rightarrow$ if system $f_1 = \dots = f_r = 0$ has no sol'n then no maximal ideal containing $I = \langle f_1, \dots, f_r \rangle$

By above fact, I must be unit ideal. //

Now prove classical form of Nullstellensatz:

Consider ring formed by inverting g. So introducing formal y s.t. $gy = 1$. i.e. work in $\mathbb{C}[\underline{x}, y]/(gy-1)$

Then the $r+1$ polynomials $f_1, \dots, f_r, g(\underline{x})y - 1$ in $\mathbb{C}[\underline{x}, y]$ have no common zero in $\mathbb{C}^{n+1}$, since $(a_1, \dots, a_n, b) \in \mathbb{C}^{n+1}$ is such that $f_1, \dots, f_r$ vanishes at $\underline{a}$ then, by assumption $g(\underline{a}) = 0$ so $g(\underline{x})y - 1 = -1$ at $(\underline{a}, b)$

By previous corollary, $\langle f_1, \dots, f_r, gy - 1 \rangle = \langle 1 \rangle$ in $\mathbb{C}[\underline{x}, y]$.

i.e. $\exists\, p_i(\underline{x}, y)$ polynomials $i = 1, \dots, r$ $q(\underline{x}, y)$ s.t. $1 = \sum_i p_i f_i + q\,(g(\underline{x})y - 1)$

In the projection to $\mathbb{C}[\underline{x},y]/(gy-1)$ (i.e. setting $y=g^{-1}$)

$$1 = \sum_i p_i(x, g^{-1}) \cdot f_i(x).$$

Clear g's in denominators of p_i's

if largest power is N,

we get $g^N = \sum r_i(x) f_i(x)$

where $r_i(x) = p_i(x, g^{-1}) g^N$.

For our situation with Riemann surfaces, have $\mathbb{C}[x,y]$ with ideal generated by single $f(x,y)$ - irreducible.

Nullstellensatz (in classical form) says if $g(x,y)$ vanishes identically on zero locus, then g^N is multiple of f for some N.

But f irreducible $\Rightarrow$ $f \mid g$.

In fact stronger result is true: Given two non-zero polynomials f, g in two vars. they have at most finitely many common zeros unless they share a factor. (can bound this finite # ~~by~~ by $\deg(f) \cdot \deg(g)$ "Bezout's bound")

pf sketch: Consider $\underbrace{\mathbb{C}(x)}_{\text{fraction field of } \mathbb{C}[x]}[y]$, the ideal $\langle f, g \rangle$ is principal in $\mathbb{C}(x)[y]$, generated by $\gcd(f,g)$.

If f, g have no common factor, then $\langle f, g\rangle$ is unit ideal in $\mathbb{C}(x)[y]$

Fact: if f, g have common factor (non-const) in $\mathbb{C}(x)[y]$ not in $\mathbb{C}(x)$, then they have a common, non-const. factor in $\mathbb{C}[x,y]$.

So suppose f,g have no common factor in $\mathbb{C}[x,y]$. Fact $\Rightarrow$ no common factor in $\mathbb{C}(x)[y]$

$\Rightarrow \langle f,g\rangle = \langle 1\rangle \Rightarrow \exists\, \underset{\in \mathbb{C}(x)[y]}{r,s}$ s.t. $rf+sg=1$.

clear denominators: $p(x) = u(x,y)f + v(x,y)g \quad u,v \in \mathbb{C}[x,y]$.

So zero of f,g is a zero of p. But p is polynomial in x alone, so has finitely many zeros.

i.e. finitely many x-values at common roots of f,g.

Symmetrically, there are only finitely many y-values at common roots.

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* Hilbert basis thm: if a ring R is Noetherian, then so is $R[x]$.

Recall Noetherian means every ideal of R is finitely generated.

(equivalently, there is no infinite strictly increasing chain of ideals $I_1 \subsetneq I_2 \subsetneq \cdots$ in R)

clearly this guarantees that every ideal is contained in maximal one.

Story is similar for projective curves. Now no non-trivial (i.e. non-const.) holom. functions defined on all of X, but quotients of ~~polys~~ homogeneous polys of same degree are meromorphic — well-defined, charts given by projections at non-zero coords. So if $X = \{ [x:y:z] \mid F(x,y,z) = 0 \}$ then if $z \neq 0$ then set $z = 1$ to obtain affine equation $f(x,y) = F(x,y,1)$

Gives chart to $\mathbb{C}^2$ and ratio of homog. polys $\dfrac{G(x,y,z)}{H(x,y,z)}$ is mapped to $g(x,y)/h(x,y)$ which is a ratio of polys with domain $(x,y) \in \mathbb{C}^2$.

where $g(x,y) := G(x,y,1)$
$h(x,y) := H(x,y,1)$

Better:

$$\left.\begin{array}{lcl} [x:y:z] \; z \neq 0 & \overset{\phi_z}{\longmapsto} & (x/z, y/z) \in \mathbb{C}^2 \\ [w_1 : w_2 : 1] & \longleftarrow & (w_1, w_2) \end{array}\right\} \text{charts for } \mathbb{P}^2(\mathbb{C})$$

So with this more proper notation $G \circ \phi_z^{-1} = g(w_1, w_2)$.

Also: projective version of nullstellensatz guarantees that $H(x,y,z)$ is identically 0 iff F divides H.

Fanciest example of Riemann surface so far: $\mathbb{P}^n(\mathbb{C})$ with set of equations which, locally, reduces to $n-1$ conditions. Place a condition on matrix of partials to be maximal rank. "Local complete intersection curves"

Similar game as above: charts from $\mathbb{P}^n(\mathbb{C}) \to \mathbb{C}^n$

$$[x_0 : \dots : x_n] \to (x_1/x_0, \dots, x_n/x_0) \mid \text{if } x_0 \neq 0$$

then use charts for smooth affine curve ((local version of charts for graph)

so charts are just projection to a coord. of $\mathbb{C}^n$ s.t. locally a graph.

i.e. local coords are ratios x_i/x_j $i \neq j$.

Definition (smooth projective curve) if X: Riemann surface in $\mathbb{P}^n(\mathbb{C})$

which is "holomorphically embedded" in $\mathbb{P}^n$ for all $p \in X$.

This means:

- For each $p \in X$, $\exists$ homog. coordinate z_j s.t.

  (a) $z_j \neq 0$ at p

  (b) for all k, z_k/z_j is holom. function in nbhd. of p

  (c) $\exists$ homog. coordinate z_i s.t. z_i/z_j is a local coord. at p.

definition is set up so that ratios of Homogeneous polys of same degree are again meromorphic functions.

indeed z_i/z_k is meromorphic since it equals $z_i/z_j \big/ z_k/z_j$, each of which is holomorphic.

thus so are rational functions in z_i/z_k, and these in turn are ratios of homogeneous polys of same degree

definition also set up so that local complete intersection curves are examples of smooth projective curves.