local properties of holomorphic maps.

Recall
$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \phi_X \downarrow & & \downarrow \phi_Y \\ \mathbb{C} & & \mathbb{C} \end{array}$$
with $\phi_Y \circ F \circ \phi_X^{-1}$ defining holomorphic function in nbhd of $\phi_X(p) =: z_0$ with local coord z

Let $\phi_Y(F(p)) =: w_0$ with local coord w

So express holomorphic map as $w = h(z)$ in nbhd. of ~~w_0~~ z_0 with $w_0 = h(z_0)$.

Expand h as Taylor series:
$$h(z) = h(z_0) + \sum_{i=m}^{\infty} c_i (z - z_0)^i \quad \text{with } c_m \neq 0. \qquad (*)$$

Here we are emphasizing the first non-zero coeff in the Taylor expansion with our indexing in sum.

All we know is $m \geq 1$.

Want to show that this integer m is an invariant of holomorphic map. at p.

call it "multiplicity". By formula $(*)$,

$\mathrm{mult}_p(F)$
$$\mathrm{mult}_p(F) = 1 + \underbrace{\mathrm{ord}_{z_0}}_{\text{denotes order of vanishing.}} (h'(z))$$

Very similar to earlier claim that lowest non-van. term in Laurent series for merom. function is independent of choice of chart.

Prove this carefully by good choice of chart.

Proposition: $F: X \to Y$ holomorphic at $p \in X$. Then $\exists$ unique $m \geq 1$
non-const.

s.t. for every chart $\phi_2 : \mathcal{U}_2 \to V_2$ on Y centered at $F(p)$,

there exists a chart $\phi_1 : \mathcal{U}_1 \to V_1$ on X centered at p,

$$\phi_2(F(\phi_1^{-1}(z))) = z^m.$$

pf: Fix a chart ϕ_2 on Y centered at $F(p)$, choose any chart $\psi: \mathcal{U} \to V$ on X centered at p. (i.e. maps p to 0)

then the holomorphic function in local coord w, call it

$$T(w) = \phi_2 \circ F \circ \psi^{-1}(w), \quad \text{has } T(0) = 0.$$

so has power series expansion

$$T(w) = \sum_{i=m}^{\infty} c_i w^i \qquad \begin{matrix}(m \geq 1) \\ c_m \neq 0\end{matrix}$$

$$= w^m \cdot S(w)$$

$S(w)$ another holomorphic function according to Taylor series with $S(0) \neq 0$.

this implies $\exists$ well-defined holom. function $R(w)$ s.t. $R(w)^m = S(w)$ in a nbhd. of 0 in w-plane

Set $\eta(w) = w R(w)$.

with $\eta'(0) \neq 0$, so η locally invertible

$\Rightarrow$ $\phi_1 := \eta \circ \psi$ is chart on X centered near p.

Just need to check uniqueness. But this follows since m can be characterized topologically as # of preimages of points in punctured nbhd of $F(p)$.

Since zeros of holom. function are isolated, and $h'(z)$ holomorphic,

then points $p \in X$ with $\text{mult}_p(F) \geq 2$ are discrete.

(All points have multiplicity ≥ 1 by definition)

Call this special points "ramification points", their images under F are "branch points"

since F open then branch points are also discrete. (Assuming F non-const here)

Examples: ⓪ $\phi : \mathcal{U} \to V \subseteq \mathbb{C}$ is chart, viewed as map from $X \to \mathbb{C}$

then $\text{mult}_p(\phi) = 1 \quad \forall \ p \in \mathcal{U}$, since ϕ bijection, and mult. records

\# of preimages.

① $F : \mathbb{C} \to \mathbb{C}$ globally defined by $z \mapsto z^m =: w$

Then only branch point is $w = 0$ (image of ramification pt. $z = 0$).

since this is only zero of derivative of z^m.

We have met "branch pts" in context of analytic continuation: Ahlfors' informal

definition that "branch pt" is point included in any branch cut

for a multi-valued function. (In this case $w \mapsto w^{1/m}$)

② X: smooth affine curve, defined by $f(x,y)$.

Any poly $g(x,y)$ restricted to X is a holomorphic function, can be regarded

as holom. map $X \to \mathbb{C}$. Simplest : $g(x,y) = x$. What are

ramification pts? Ans: $p = (x_0, y_0)$ s.t. $\partial f / \partial y \big|_p = 0$

Indeed if $\partial f / \partial y \big|_p \neq 0$ then x is a chart, so has $\text{mult}_p(x) = 1$ at p (i.e. not ramified)

if $\partial f / \partial y \big|_p = 0$, then must have $\partial f / \partial x \big|_p \neq 0$ since X smooth so

near p, X locally graph of form $(g(y), y)$ some g holom.

$\Rightarrow$ $f(g(y), y)$ vanishes identically in nbhd of y_0 with $p = (x_0, y_0)$.

$\Rightarrow$ $\left(\frac{\partial f}{\partial x} \cdot g'(y) + \frac{\partial f}{\partial y}\right)\Big|_{(g(y),y)}$ identically 0 in nbhd of y_0.

But then evaluating at $p = (x_0, y_0)$, we assumed $\partial f/\partial y|_p = 0$ $\partial f/\partial x|_p \neq 0$

$\Rightarrow$ $g'(y_0) = 0$

But when $x = g(y)$, this is precisely the criterion for the map x to be ramified.

Similar statement can be found in Miranda's book for projective curves.

(3) f: meromorphic function on Riemann surface X
viewed as holom. map $X \to \mathbb{C}_\infty$ (call it F)

<u>Cases</u>: $p \in X$ is not a pole. Then $z_0 = f(p)$
so that $\text{mult}_p(F) = \text{ord}_p(f - f(p))$

$p \in X$ is a pole. Then $\text{ord}_p(f)$ negative, p is zero of $1/f$.
$\text{mult}_p(F) = -\text{ord}_p(f)$ (use other chart.)