Recap: we've been studying Riemann surfaces.

- collection of compatible local topological isomorphisms to open sets of \( \mathbb{C} \)
- holomorphic \( \phi \)
- "charts"

Study holomorphic functions / meromorphic functions

\[
F : X \rightarrow \mathbb{C}
\]

Local theory is same as that for

- open mapping theorem
- identity theorem (agrees on limit pt)
- discreteness of preimages under \( F: \text{holom.} \)

On Friday, began to study invariants.

Multiplicity of \( F \) at \( p \in X \):

- order of vanishing of Taylor series
  at \( \phi_x(p) = z_0 \) in coord \((z - z_0)\).
- as expansion \((w - w_0)\) (with \( z_0 \mapsto w_0 \))

Under \( \phi_y \circ F \circ \phi_x^{-1} \)

Multiplicity is "generically" equal to 1. \((\forall p : \text{mult}_p(F) > 1\) is discrete in \( X \))

E.g. \( z \mapsto z^2 \) is holomorphic map. \( \text{mult.} = 3 \) at \( z = 0 \) and \( = 1 \) elsewhere.

B/c \( \text{mult.} \) at \( z = 5 \): \( W = W_0 \)

\[
\frac{5^3}{5^3} = \sum_{n=1}^{3} a_n (z-5)^n \quad a_n = \frac{f^{(n)}(5)}{n!} \quad f^{(1)}(5) \neq 0
\]
Example 2: \( f \): meromorphic function on \( X \), then let \( F \) be associated holomorphic map:

\[
F(p) = \begin{cases} 
  f(p) & \text{if } p \text{ not pole} \\
  \infty & \text{if } p \text{ pole}
\end{cases}
\]

If \( p \) not a pole, then \( \text{mult}_p(F) = \text{ord}_p(f - f(p)) \).

In particular if \( p \) is a zero, \( \text{mult}_p(F) = \text{ord}_p(f) \).

If \( p \) is a pole, use chart from stere. proj. from south pole.

or map \( \mathbb{S} \to \mathbb{S} \) equivalently. Get \( \text{mult}_p(F) = -\text{ord}_p(f) \).

(think in terms of chart centered at origin)

Use the local invariant to make global one — degree \( (F) \), \( F: X \to Y \) holom. \( X, Y \) compact.

Given any \( y \in Y \), consider \( \sum_{p \in F^{-1}(y)} \text{mult}_p(F) \).

If \( X, Y \) compact, then \( F^{-1}(y) \) finite set, so sum is well defined.

Proposition: This sum is a fixed constant, independent of \( y \in Y \). (called deg \((F)\))

If \( y \in Y \), show that \( y \mapsto \sum_{p \in F^{-1}(y)} \text{mult}_p(F) \) is locally constant function.

Since \( Y \) connected, then must be constant function. i.e., for every \( y \in Y \)

function is constant.

Lemma: if \( F^{-1}(y) = \{x_1, \ldots, x_n \} \in X \), then

- if \( y' \) near \( y \), \( F^{-1}(y') \) contained in nbhd of \( x_i \).

Proof of Lemma: if \( y' \) arbitrarily close to \( y \) whose precimages under \( F \) are not

all contained in nbhd of \( x_i \). Then construct sequence of \( x' \)'s outside nbhd of \( x_i \)

whose images under \( F \) converge to \( y \).
Since $X$ compact, can extract a convergent subsequence $p_n \rightarrow y$ in $X$ with $p_n \rightarrow x \in X$, some $x$, \( \lim F(p_n) = y \). But since $F$ continuous, must have $F(x) = y$. But this is a contradiction since then $x \in \{ x_1, ..., x_n \}$ and is limit pt of $p_n$'s which lie outside all nbhds of $x$.

So to analyze whether our sum is locally constant, we can use charts for fixed $y$ and $F^{-1}(y) = \{ x_1, ..., x_n \}$.

Here we've seen that we can plot little charts: centered at $x_i$ and at $y$.

and of form $W = z_i^{m_i}$ ($z_i$ : local coord for $x_i$) $m_i$ : some integer $\geq 1$.

But each of $z_i \rightarrow z_i^{m_i}$ has property that $\deg (z_i \rightarrow z_i^{m_i})$ is locally const.

so in total $\deg (F) = \sum m_i$.

and we're done.

Corollary: $X$: compact Riemann surface $f$: meromorphic with single \qquad \text{from } X \rightarrow C^\infty \text{ as}$

$\text{ on } X$ \begin{itemize}
$\text{ simple pole}$
\end{itemize}

$C^\infty$: \qquad $f$: single simple pole \Rightarrow \text{ if } F: X \rightarrow C^\infty \text{ corresponds to } f$

\text{ then } $\deg (F) = - \text{ord}_p (f)$ where $p$: pole.

$= 1$.

But degree one map is 1-1, so must have isomorphism

(earlier we proved non-constant holomorphic map $f: X \rightarrow Y$ is onto

if $X$ compact)
Proposition: \( f \) non-constant meromorphic function on \( X \) compact. Then
\[
\sum_{p} \operatorname{ord}_{p}(f) = 0.
\]

Let \( F \) be the associated holomorphic map \( X \to \mathbb{C} \).

\( z_i \) : pts. of \( X \) mapping to \( 0 \) (zeros)
\( p_i \) : pts. of \( X \) mapping to \( \infty \) (poles)

\[
\deg(F) = \sum_{i} \operatorname{mult}_{z_i}(F) = \sum_{j} \operatorname{mult}_{p_i}(F)
\]
\[
= \sum_{i} \operatorname{ord}_{z_i}(F) = -\sum_{j} \operatorname{ord}_{p_j}(F)
\]

But
\[
\sum_{p} \operatorname{ord}_{p}(F) = \sum_{i} \operatorname{ord}_{z_i}(F) + \sum_{j} \operatorname{ord}_{p_j}(F)
\]
\[
= \deg(F) - \deg(F) = 0.
\]

Previously asserted this for Riemann sphere using that all meromorphic functions were rational functions. (i.e. used strong characterization)

Next: relate degree to topology through genus. Discuss topology basics for a bit. Often use language of simplicial complexes:

\[
\text{simplices: } \quad 0 \quad 1 \quad 2 \quad 3 \quad \ldots \quad (\text{has general coord. definition } \sum_{i=1}^{n} x_i = 1, \ x_i > 0)
\]

\[
\text{simplicial complex: collection of simplices glued so that intersection of any two } 6_1, 6_2 \text{ is a face of both } 6_1, 6_2 \quad \text{e.g. Hatcher Ch.2}
\]

Euler using them to construct polyhedra, later used to study manifolds via homeomorphisms from simplices to \( X \) with compatibility properties.