On Friday, we were exploring genus of compact Riemann surfaces, cut out of \( \mathbb{P}^2(\mathbb{C}) \) by homogeneous polynomials of small degree.

Linear: \( S^2 \cong \mathbb{P}^1(\mathbb{C}) \)

Quadratic: \( F(x,y,z) = v^T \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} v \) with \( v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \)

We had shown \( X_F: R.S. \) made from \( F \cong S^2 \) if we could show that \( A_F \cong I_3: 3 \times 3 \) identity matrix with \( A \cong B \) if \( \exists C \) such that \( C^T A C = B \).

(This relation says, since \( A \) arbitrary, that all \( R.S. \) defined by quad. form are isomorphic. Showed \( F(x,y,z) = x^2 - y^2 \) is isom. to \( S^2 \).)

Left to show \( A = C^T C \) with \( C \) invertible.

Error from Friday: Can't use spectral theorem that allows us to diagonalize. For \( \mathbb{C} \). matrices, \( A \) requires \( A = A^* \).

Can't use \( A = U^T D U \) with \( U \) upper triangular.

Since this requires non-vanishing of minors.

So followed Clemens: if \( A \) not \( 0 \)-matrix, then can find \( V \)

\[ \langle v_1, v_1 \rangle = 1 \] where \( \langle v_1, v_1 \rangle = v_1^T A v_1 \)

Find invertible matrix \( L \) s.t. \( L(v_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) and so that \( L^T A L \) has form:

\[ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{sym} & \text{sym} \\ 0 & \text{sym} & \text{sym} \end{pmatrix} \]

Repeat. Either \( d_1 e_1 f_1 = 0 \) or \( \exists v_2 \)

with \( \langle v_2, v_2 \rangle = 1 \), make change of vars. Arrive at \( C \).

Find that \( A \)'s grouped into equivalence classes by rank. (# of 1's on diagonal).

Since \( A \) invertible, has full rank.
Fact: Any symmetric $A$ is expressible as $T^T T = A$ for some invertible $T$.

(A is diagonalizable, write as $A = D C D^{-1}$ with $D$ diagonal, composed of eigenvalues. $\lambda_i$.

If $A$ symmetric, then can write $A = U^T D U$ with $U$ lower triangular, $U^T$ upper triangular

And $CC^T = I$ so get $A = \begin{pmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & \cdots \end{pmatrix}$

If $A$ symmetric, then can write $A = U^T D U$

That is, the Riemann surface for any symmetric matrix $A$ is isomorphic to the identity matrix, that is they are all isomorphic.

Just pick convenient choice of $F$ to study gens.

$F = x^2 - y^2$ is non-singular. Then have iso. $U^T \sim X F$

Check easily this is holomorphic map.

More clever: Reconstruct compact Riemann

More generally: $X, Y$ R.S. with open sets $U \subset X$, $V \subset Y$ and homeom. $\phi: U \to V$, then "glue" $X$ and $Y$ by forming $\bar{Z} = X \sqcup U \rightarrow Z = \{ x \in X - U, y \in Y - V, (x, \phi(x)) \text{ if } x \in U \}$

Prototype: Riemann sphere: obtained by gluing two copies of $C$

( had two charts: proj. from north/south poles which mapped onto $C$ )

overlap $C^*$ with transition map $\phi(z) = 1/z$.

identify points in overlap according to $\phi$.
topology on $\mathcal{Z}$: quotient topology from $\pi: X \sqcup Y \to X \sqcup Y / \phi \cong \mathcal{Z}$

so $\Omega \in \mathcal{Z}$ is open iff $\pi^{-1}(\Omega)$ open in $X \sqcup Y$.

(this of course works for $X \sqcup Y$ topological spaces, but ...)

We can further put complex structure on $\mathcal{Z}$ if $\phi: U \to V$ is

isomorphism of Riemann surfaces.

Prop: $U \subset X, V \subset Y$ R.S. $\phi: U \to V$ isom.

exists structure on $\mathcal{Z} = X \sqcup Y / \phi$ s.t. inclusions of $X, Y$ into $\mathcal{Z}$ are holomorphic maps. (Resulting $\mathcal{Z}$ is conn., not nec. Hausdorff. If Hausdorff, then $\mathcal{Z}$ is a R.S.)

Let $j_x: X \to \mathcal{Z}$ be natural inclusions. Given chart $\psi: U_d \to \psi(U_d)$ on $X$

then $j_x(U_d)$ is open in quotient topology

( the set $U_d = U_d \cap (X-U) \cup U_d \cap U$

so $\pi^{-1}(j_x(U_d)) = U_d \sqcup \phi(U_d \cap U)$

charts: $\psi_x \circ j_x^{-1}$ with $\psi_x$: chart on $X$

$\psi_y \circ j_y^{-1}$ with $\psi_y$: chart on $Y$

homeomorphisms which cover $\mathcal{Z}$. Check compatibility, which is easy since

charts agree only on identified $U$ and $V$, related by isom. $\phi$.

So reduces to compatibility of original charts.

Moreover, each of these maps must be

charts if $j_x, j_y$ are to be holomorphic.
Now use the giving principle for affine curves.

Consider smooth affine curve given by

\[ X = \{ (x,y) \mid y^2 = h(x) \} \quad \text{h has degree } 2g+1 \text{ or } 2g+2, \text{ distinct roots.} \]

\[ U = \{ (x,y) \in X \mid x \neq 0 \} \]

\[ Y = \{ (z,w) \mid w^2 = k(z) \} \quad k(z) := z^{2g+2} h(z) \quad \text{(poly. in } z \text{ with distinct roots)} \]

\[ V = \{ (z,w) \in Y \mid z \neq 0 \} \]

\[ \phi : U \to V \quad \text{isomorphism} \]

\[ (x,y) \mapsto (z,w) := \left( \frac{1}{y}, \sqrt[2g+1]{x} \right) \]

\[ (\frac{1}{z}, \frac{1}{w^{2g+1}}) \leftrightarrow (z, w) \]

Claim: \[ \mathbb{Z} := X \amalg Y / \phi \]

is compact Riemann surface of genus \( g \).

(remember \( g \) appeared in the degree of the affine curves \( h, k \) defining \( x, y ) \)

Proof of claim: compactness follows since \( \mathbb{Z} \) is union of compact sets (viewed as subsets of \( \mathbb{C} \) via inclusion)

\[ \mathbb{Z} = \{ (x,y) \in X \mid \|x\| \leq 1 \} \quad \text{and} \quad \mathbb{Z} = \{ (z,w) \in Y \mid \|z\| \leq 1 \} \]

Calculate genus using Horwitz formula, as follows:

\[ X = \{ (x,y) \mid y^2 = h(x) \} \]

has \( \alpha_0 \) map, proj. to \( X \), which is holomorphic. Extend this to a map \( \pi : \mathbb{Z} \to \mathbb{C}_0 \) (defined at points in \( Y \setminus V = \{ (0, w) \mid w^2 = k(0) \} \) so that \( \pi \) continuous.

( holomorphic map as meromorphic function)

\[ \deg(\pi) = 2 \] since \( y^2 = c \) has 2 solns if \( c \neq 0 \).

Branch points are zeros (i.e. roots) of \( h(x) \) and \( 0 \) if \( \deg(h(x)) \) odd.
Thus, for either case, have $2g + 2$ points with multiplicity 2. These appear in "error" term of Hurwitz formula.

$$-\chi(z) = \deg(\pi) \cdot (-\chi(C_\infty)) + \frac{\text{error}}{2g+2}$$

$$\chi(z) = -4 + 2g + 2 = 2g - 2 \quad \text{i.e. genus of } \mathcal{Z} \text{ is } g.$$

- Meromorphic functions on hyperelliptic Riemann surfaces.

  Described similarly to meromorphic functions on elliptic curve $C/\Lambda$.

  These we broke up elliptic functions into even, odd, used $g, g'$

  Here introduce similar involution (order 2 automorphism)

  of $\mathcal{Z}$:

  $6: \mathcal{Z} \to \mathcal{Z}$

  taking $(x, y) \in \mathcal{Z} \mapsto (x, -y)$

  $(z, w) \in \mathcal{Z} \mapsto (z, -w)$

  It is a holomorphic map on $\mathcal{Z}$, so given merom. $f$ on $\mathcal{Z}$

  then $6^*f := f \circ 6$ is merom. on $\mathcal{Z}$.

  And $f + 6^*f$ is $6^*$-invariant, since $6^2 = \text{id}$.

  Notice that projection $\pi: \mathcal{Z} \to C_\infty$ commutes with $6$: $\pi \circ 6 = \pi$.

  So basic example of $6^*$ invariant function is pullback under $\pi$ of meromorphic function $r$ (for "rational")

  on $C_\infty$.

  Lemma: $g$ merom. on $\mathcal{Z}$ s.t. $6^*g = g$. Then $\exists f! r \in C_\infty$

  s.t. $g = r \circ \pi$. 