Thus, for either case, have $2g+2$ points with multiplicity 2.
$2g+1$ or $2g+2$

these appear in "error" term of Hurwitz formula.

$$-\chi(Z) = \deg(\pi)\cdot(-\chi(\mathbb{C}_\infty)) + \underbrace{\text{error}}_{2g+2}$$

$$\chi(Z) = -4 + 2g+2 = 2g-2 \quad \text{i.e. genus of } Z \text{ is } g.$$

Meromorphic functions on hyperelliptic Riemann surfaces.

Described similarly to merom. functions on elliptic curve $\mathbb{C}/\Lambda$

There we broke up elliptic functions into even, odd, used $\wp, \wp'$ to describe resulting pieces.

Here introduce similar involution (order 2 automorphism) of Z:

$$\sigma : Z \to Z$$

taking $(x,y) \in X \mapsto (x,-y)$
$(z,w) \in Y \mapsto (z,-w)$

It is a holomorphic map on Z, so given merom. f on Z

then $\sigma^* f := f \circ \sigma$ is merom. on Z.

And $f + \sigma^* f$ is σ^*-invariant, since $\sigma^2 = \mathrm{id}$.

Notice that projection $\pi : Z \to \mathbb{C}_\infty$ commutes with σ: $\pi \circ \sigma = \pi$

so basic example of σ^* invariant function is pullback under π of meromorphic function r (for "rational") on $\mathbb{C}_\infty$.

<u>Lemma</u>: g merom. on Z s.t. $\sigma^* g = g$. Then $\exists!\ r \in \mathbb{C}_\infty$ s.t. $g = r \circ \pi$.

pf. of lemma: Given g, let $r(p) := g(q)$ where q is preimage of p under π (i.e. $\pi(q) = p$.)

π has deg. 2 but since g is σ^* invariant, this is well-defined no matter the choice of preimage.

Check that r is meromorphic on $\mathbb{C}_\infty$ since g meromorphic on Z, π holomorphic. //

Write any meromorphic function f on Z as $f = f^+ + f^-$ where

$$f^+ = \frac{f + \sigma^*(f)}{2} \qquad f^- = \frac{f - \sigma^*(f)}{2}$$

But the projection to y on Z has $\sigma^* y = -y$

so all anti-invariant merom. functions on Z are of form $y \cdot s$, with s rational in x.

i.e. merom. on Z have unique rep'n as

$$f = r(x) + y\,s(x)$$ r, s rational functions of x.

More topology review — X, Y connected topological spaces (see Miranda · III.4)

① $F : X \to Y$ is a "local homeomorphism" if for each $x \in X$, $\exists$ open nbhd $\mathcal{U}$ s.t. $F|_{\mathcal{U}}$ is homeomorphism to $F(\mathcal{U}) \subseteq Y$.

② $F : X \to Y$ is a "covering map" if for each $y \in Y$, $\exists$ open nbhd V s.t. $F^{-1}(V)$ is disjoint union of open sets $\mathcal{U}_\alpha$ in X with $F|_{\mathcal{U}_\alpha}$ a homeomorphism from $\mathcal{U}_\alpha$ to V.

So a covering map is a local homeomorphism. If F proper (inverse image of compacta are compact) then proper local homeomorphism $\iff$ finite covering map.

(finite : $F^{-1}(y)$ is finite set $\forall\, y \in Y$)

Basic relation between covering maps and fundamental gps.

Given topological space Y with distinguished point y_0 "base point" define fundamental gp. $\pi_1(Y, y_0) :=$ homotopy classes of loops based at y_0.

~~[illegible]~~

Examples : $Y = \mathbb{C}, \mathbb{D}$ then $\pi_1(Y)$ trivial

$Y = \mathbb{C} \setminus \{0\}$, then $\pi_1(Y) \cong \mathbb{Z}$ or $Y = \mathbb{C} \setminus \{z_1, \dots, z_n\}$ then $\pi_1(Y) \cong$ free gp. on n generators

$Y = \mathbb{C}/M$: torus then $\pi_1(Y) = \mathbb{Z} \times \mathbb{Z}$

$Y =$ genus g compact top. space, $\pi_1(Y) = \langle \underbrace{a_1, b_1, \dots, a_g, b_g}_{2g \text{ gens}} \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$

where $[a, b] = aba^{-1}b^{-1}$ commutator.

If Y has "reasonable" local structure (for us, with Riemann surfaces our local structure is disc!)

e.g. locally path connected
and $\underbrace{\text{semi-locally}}$ simply conn'd.
loops in nbhd $\mathcal{U}$ are contractible.
but contractions can occur in Y.
(not just restricted to $\mathcal{U}$)

then there is a 1-1 correspondence between:

- equivalence classes of coverings $F: X \to Y$, X: connected.
(so data here is pair $(X, F: X \to Y)$)
- conjugacy classes of subgps of $\pi_1(Y, y_0)$.

[Recall, two coverings $F: X \to Y$, $F': X' \to Y$ are equivalent if $\exists$ homeomorphism $\phi: X \to X'$ such that $F = F' \circ \phi$]. (See Section 1.3 in Hatcher)

Correspondence is given as follows: Given $F: X \to Y$, there is an induced homomorphism

$$F_*: \pi_1(X, x_0) \to \pi_1(Y, F(x_0))$$
$$[\gamma] \mapsto [F \circ \gamma]$$

for $\pi_1(Y, y_0)$ pick $x_0 \in F^{-1}(y_0)$ and then subgp. for $F: X \to Y$ is just image $F_*(\pi_1(X, x_0))$. Different choices of $x_0 \in F^{-1}(y_0)$ give different conjugates of this subgp in $\pi_1(Y)$.

Now want to show that a covering can be constructed for any subgp. of $\pi_1(Y)$.

E.g. trivial subgp. $G: \tilde{Y} \to Y$ with $\pi_1(\tilde{Y})$ trivial.
"universal cover"

$$\tilde{Y} = \{(y, A) \mid y \in Y, A: \text{homotopy class of paths in } Y \text{ from } y_0 \text{ to } y\}$$

$G((y, A)) = y$. Need to check that give $\tilde{Y}$ topology so that G is covering ~~space~~ map.