Thus, for either case, have \( 2g+2 \) points with multiplicity 2.

These appear in "error" term of Hurwitz formula.

\[-X(\mathbb{Z}) = \deg(\pi) \cdot (-X(C_0)) + \frac{\text{error}}{2g+2}\]

\[X(\mathbb{Z}) = -4 + 2g+2 = 2g-2 \quad \text{i.e. genus of } \mathbb{Z} \text{ is } g.

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Meromorphic functions on hyperelliptic Riemann surfaces.

Described similarly to meromorphic functions on elliptic curve \( C/\Lambda \).

There we broke up elliptic functions into even, odd, used \( \theta, \theta' \) to describe resulting pieces.

Here introduce similar involution (order 2 automorphism)

of \( \mathbb{Z} \):

\[b : \mathbb{Z} \to \mathbb{Z}\]

taking \((x,y) \in \mathbb{X} \leftrightarrow (x,-y)\)

\[(z,w) \in \mathbb{Y} \leftrightarrow (z,-w)\]

It is a holomorphic map on \( \mathbb{Z} \), so given merom. \( f \) on \( \mathbb{Z} \)

then \( b^*f := f \circ b \) is merom. on \( \mathbb{Z} \).

And \( f + b^*f \) is \( b^* \)-invariant, since \( b^2 = \text{id} \).

Notice that projection \( \pi : \mathbb{Z} \to C_0 \) commutes with \( b : \pi \circ b = \pi \)

so basic example of \( b^* \) invariant function is

pullback under \( \pi \) of meromorphic function \( r \) (for "rational")
on \( C_0 \).

Lemma: \( g \) merom. on \( \mathbb{Z} \) s.t. \( b^*g = g \). Then \( F! r \in C_0 \)
s.t. \( g = r \circ \pi \).
pf. of lemma: Given \( g \), let \( r(p) = g(q) \) where

\( q \) is preimage of \( p \) under \( \pi \) (i.e. \( \pi(q) = p \) - \( \pi \) has deg. 2

but since \( g \) is \( 6^* \) invariant, this is well-defined no matter the choice of preimage.

Check that \( r \) is meromorphic on \( \mathbb{C} \).

since \( g \) meromorphic on \( \mathbb{A} \)

\( \pi \) holomorphic. \( \checkmark \)

---

Write any meromorphic function \( f \) on \( \mathbb{A} \)
as \( f = f^+ + f^- \) where

\[
\begin{align*}
f^+ &= f + \frac{6^*(f)}{2} \\
f^- &= f - \frac{6^*(f)}{2}
\end{align*}
\]

But the projection to \( y \) on \( \mathbb{A} \) has \( 6^*y = -y \)
so all anti-invariant merom. functions on \( \mathbb{A} \) are of form \( y \)’s, with \( s \) rational in \( x \).

i.e. merom. on \( \mathbb{A} \) have unique rep’ as

\[
f = r(x) + y s(x) \quad r, s \text{ rational functions of } x.
\]
More topology review — \( X, Y \) topological spaces (see Miranda III.4)

1. \( F: X \to Y \) is a "local homeomorphism" if for each \( x \in X \), \( \exists \text{ open nbhd } U \)
   
   s.t. \( F|_U \) is homeomorphism \( \iff F(U) \subseteq Y \).

2. \( F: X \to Y \) is a "covering map" if for each \( y \in Y \), \( \exists \text{ open nbhd } V \)
   
   s.t. \( F^{-1}(V) \) is disjoint union of open sets \( U_x \) in \( X \) with \( F|_{U_x} \) a homeomorphism from \( U_x \) to \( V \).

So a covering map is a local homeomorphism. If \( F \) proper (inverse image of compacta are compact)

then proper local homeomorphism \( \iff \) finite covering map.

(finite: \( F^{-1}(y) \) is finite set \( \forall y \in Y \))

Basic relation between covering maps and fundamental gps.

Given topological space \( Y \) with distinguished point \( y_0 \) "base point"

define fundamental gp: \( \pi_1(Y, y_0) := \) homotopy classes of loops

based at \( y_0 \).

Examples: \( Y = \mathbb{C}, D \) then \( \pi_1(Y) \) trivial

\( Y = \mathbb{C} \setminus \{0\} \), then \( \pi_1(Y) = \mathbb{Z} \) or \( Y = \mathbb{C} \setminus \{1, 2, \ldots, n\} \)

then \( \pi_1(Y) \) is free gp. on \( n \) generators

\( Y = \mathbb{C}/\mathbb{Z} \) torus then \( \pi_1(Y) = \mathbb{Z} \times \mathbb{Z} \)

\( Y = \) genus \( g \) compact topo. space, \( \pi_1(Y) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] 
\]

\(~2g \text{ gens}\)

\( a b a^{-1} b^{-1} \)

commutator.
If $Y$ has "reasonable" local structure (for us, with Riemann surfaces or local structure is direct!)

e.g. locally path connected
and semi-locally simply conned.
loops in nbhd $U$ are contractible.
but contractions can occur in $Y$.
(not just restricted to $U$)

Then there is a 1-1 correspondence between:

- equivalence classes of covering $F : X \to Y$, $X$ connected.
  (so data here is pair $(X, F : X \to Y)$)

- conjugacy classes of subgps of $\pi_1(Y, y_0)$.

[Recall, two coverings $F : X \to Y$, $F' : X' \to Y$ are equivalent if

there is homeomorphism $\phi : X \to X'$ such that $F = F' \circ \phi$].

(See Section 1.3 in Hatcher)

Correspondence is given as follows: Given $F : X \to Y$, there is an induced

homomorphism $F_* : \pi_1(X, x_0) \to \pi_1(Y, F(x_0))$

$[\gamma] \mapsto [F \circ \gamma]$

for $\pi_1(Y, y_0)$ pick $x_0 \in F^{-1}(y_0)$ and then subgp. for $F : X \to Y$

is just image $F_*([\pi_1(X, x_0)])$. Different choices of $x_0 \in F^{-1}(y_0)$
give different conjugates of this subgp in $\pi_1(Y)$.

Now want to show that a covering can be constructed for any subgp. of $\pi_1(Y)$.

E.g. trivial subgp. $G : \tilde{Y} \to Y$ with $\pi_1(\tilde{Y})$ trivial.

"universal cover" $\tilde{Y} = \{ (y, A) \mid y \in Y, A : homotopy class of paths in $Y$

from $y_0$ to $y$

$G((y, A)) = y$. Need to check that $G$ is a covering map.