Proposition : (i) $f : \Omega_1 \to \Omega_2$ conformal, bijective, then

$f^{-1} : \Omega_2 \to \Omega_1$ is conformal.

(ii) $f : \Omega_1 \to \Omega_2$, $g : \Omega_2 \to \Omega_3$ conformal. bijective, then $g \circ f$ is confomal, bijective.

pf : immediate. (i) f^{-1} exists since f bijective. Inverse function thm $\Rightarrow$

f^{-1} analytic with $\frac{d}{dw}(f^{-1}) = 1 / \frac{d}{dz}(f)$

with $f(z) = w$.

(so in particular $\frac{d}{dw}(f^{-1}) \neq 0$ so f^{-1} conformal)

(and defined when f conformal)

(ii) compositions of analytic bijections are analytic bijections. chain rule gives non-zero deriv.

(so in particular, bijective conformal maps of Ω to itself forms gp.)

Proposition 2 : u harmonic on Ω_2, $f : \Omega_1 \to \Omega_2$ analytic, then $u \circ f$ harmonic on Ω_1

pf : Pick $z \in \Omega_1$ $w = f(z)$. $\mathcal{U}$: nbhd of w in Ω_2

$V := f^{-1}(\mathcal{U})$: nbhd. of z.

Want to show $u \circ f$ harmonic on V. (suffices, since being harmonic is local condition)

Here we use that u : harmonic $\Rightarrow$ $\exists\, g$ on $\mathcal{U}$ s.t. $u = \mathrm{Re}(g)$.

Then $u \circ f = \mathrm{Re}(g \circ f)$ but $g \circ f$ analytic so $\mathrm{Re}(g \circ f)$ harmonic.

Riemann mapping theorem: Given any simply conn. region Ω not whole plane,

and $z_0 \in \Omega$, $\exists!\ f$, analytic on Ω, s.t.

$f : \Omega \to B(0;1)$ open unit ball bijective

(normalized so that $f(z_0) = 0$, $f'(z_0) > 0$)

uniqueness: Ahlfors: f_1, f_2 two such maps, then

$f_1 f_2^{-1}$ is one-one conformal map of $B(0;1)$ onto itself.

Only such maps are linear fractional transformations.

(Schwarz' lemma) Recall this in class.

Normalizations imply $S := f_1 \circ f_2^{-1}$ with $S(0) = 0$, $S'(0) > 0$

must be $S(w) = w$

i.e. $f_1 = f_2$.

Corollary: Any two simply conn. regions $A, B \neq \mathbb{C}$ are "conformally" equivalent": there exists conformal map

$f: A \to B$.

pf: $f_1 : A \to D$ set $f = f_2^{-1} \circ f_1$

$f_2 : B \to D$

existence $\mathcal{F}$: family of functions g ~~~~ defined on Ω s.t.

(i) g is analytic and one-one

(ii) $|g(z)| \le 1$ in Ω

(iii) $g(z_0) = 0$, $g'(z_0) > 0$.

show $\mathcal{F}$ is non-empty, $\exists$ member f with maximal derivative, at z_0.

and finally this maximal f is desired function (i.e. surjective on $B(0;1)$)

(A) $\mathcal{F}$ is non-empty.

Pick $a \notin \Omega$, define single valued branch of $\underbrace{\sqrt{z-a}}_{h(z)}$ on Ω.

possible since Ω simply connected.

Claim: $h(z)$ is one-one, and in fact never have $h(z_1) = \pm h(z_2)$ for $z_1, z_2 \in \Omega$.

$h(z_1) = h(z_2) \Rightarrow z_1 = a + h(z_1)^2 = a + h(z_2)^2 = z_2$

$h(\Omega)$, $-h(\Omega)$ disjoint.

Consider $\Omega \xrightarrow{h} h(\Omega)$ $\Rightarrow$ $|h(z) + h(z_0)| \geq \rho$

$$h(\Omega) \supset \{|w - h(z_0)| < \rho\}$$

$\Rightarrow 2|h(z_0)| \geq \rho$

for some ρ.

(so doesn't meet disk $|w + h(z_0)| < \rho$. by claim)

then
$$g_0(z) := \frac{\rho}{4} \underbrace{\frac{|h'(z_0)|}{|h(z_0)|^2} \cdot \frac{h(z_0)}{h'(z_0)}}_{\text{const.}} \left[\frac{h(z) - h(z_0)}{h(z) + h(z_0)} \right] \in \mathcal{F}.$$

h one-one $\Rightarrow$ g_0 (h composed with linear map) one-one.

$$g(z_0) = 0. \qquad g_0'(z_0) = \rho/8 \cdot \frac{|h'(z_0)|}{|h(z_0)|^2} > 0.$$

Finally,
$$\left| \frac{h(z_0) - h(z_0)}{h(z) + h(z_0)} \right| = |h(z_0)| \cdot \underbrace{\left| \frac{1}{h(z_0)} - \frac{2}{h(z) + h(z_0)} \right|}_{\leq 4/\rho}$$

using triangle inequality.

So $|g_0(z)| \leq 1$ on Ω.

(B) $\exists$ $f \in \mathcal{F}$ with maximal derivative at z_0.

$\{ g'(z_0) \}_{g \in \mathcal{F}}$ has least upper bound B (might $= \infty$)

Pick sequence $g_n \in \mathcal{F}$ s.t. $\lim g_n'(z_0) = B$.

<u>Claim</u>: $\exists$ subsequence $\{ g_{n_k} \}_k \longrightarrow f$: analytic function uniformly on compact sets

pf of <u>claim</u>: Thm. of Normal Families.