On Monday, exploring the covering map

$$X \setminus F^{-1}(B) \longrightarrow Y \setminus B$$

corresponding to holom. map $F: X \to Y$ of degree d. This has associated "monodromy rep'n" – a gp. homom. $\rho : \pi_1(Y \setminus B) \to S_d$

Abstractly: subgroups of index d in $\pi_1(Y \setminus B)$ $\xleftrightarrow{1-1}$ transitive perm. rep'ns ρ on d letters

$$H = \{ [\gamma] : \rho([\gamma])(1) = 1 \} \longleftarrow\!\shortmid \rho$$

$$H \longmapsto \pi_1 \text{ acts on } \pi_1/H$$

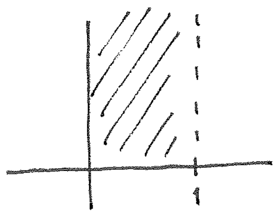
Geometrically: lift loop $[\gamma]$ under F^{-1} and consider what happens to preimages as we traverse around in loop. How do they permute?

Slightly more generally:

$Y \setminus B$: punctured unit disk. $\pi_1(Y \setminus B) \cong \mathbb{Z}$ generated by loop with winding number 1.

Consider $z \mapsto e^{2\pi i z}$

then



$[0,1)$ maps to unit circle.

and strip $\mathrm{Im}(z) > 0$, $0 \le \mathrm{Re}(z) < 1$ maps to unit disk $\setminus$ origin.

(isom. of Riemann surfaces)

and $\mathcal{H}$: upper half plane $= \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$ is universal cover

where loop around origin lifts to horizontal line segment of integer length

so $\pi_1(Y \setminus B)$ acts on $\mathcal{H}$ by translation by n where n: winding # of loop in $\pi_1(Y \setminus B)$

$(z \mapsto z+n)$

Subgps of $\mathbb{Z}$ are of form $N\mathbb{Z}$ with $N \geq 0$.

and $N = 0 \longleftrightarrow \mathcal{H}$

$N = 1$ (so subgp. is $\mathbb{Z}$) $\longleftrightarrow$ punctured disk

what if $N \geq 2$? quotient $\mathcal{H}$ by $z \mapsto z+N$ gives strip of width N.

Another punctured disk (Miranda's notation: D_N)

$$\mathcal{H} \longmapsto D_N = \{ |z| < 1, z \neq 0 \}$$

$$z \longmapsto e^{2\pi i z/N} = w_N$$

$$D_N \longrightarrow D_1$$

$$w_N \longmapsto w_N^N = w_1$$

So N-th power maps are intermediate covering maps between punctured disk D_1 and its universal cover $\mathcal{H}$.

What is monodromy rep'n for cover $D_N \to D_1$?

$w_N \mapsto w_N^N =: w_1$

Just need to know effect of generator of $\pi_1(D_1)$ = loop of winding # 1.

loop $\gamma = c e^{2\pi i t}$ $t \in [0,1]$, c small real so that loop is in unit disk.

based at c, with preimages $c^{1/N} \xi^i$ with $\xi = e^{2\pi i/N}$

The loop $\gamma = c e^{2\pi i t}$ lifts to $\tilde{\gamma}_i = \xi^i e^{2\pi i t/N} c^{1/N}$ which starts at $c^{1/N} \xi^i$ and ends at $c^{1/N} \xi^{i+1}$

$i = 1, \dots, N$

conclusion: $1 \in \mathbb{Z} \cong \pi_1(D_1) \longmapsto$ permutation σ $(1\, 2 \cdots n)$ "n-cycle" mapping each $i \mapsto i+1$.

This example is all we need to know, remembering that locally, holomorphic maps are $z \mapsto z^m$.

Indeed given a branch point $b \in Y$, consider punctured nbhd. $W \setminus \{b\}$. If $F: X \to Y$ has degree d, then b has $< d$ preimages in X, call this integer k. $\{u_1, \dots, u_k\} = F^{-1}(b)$. with open nbhds $U_1, \dots U_k$. Can choose W small enough so that U_i are disjoint. We may choose z_j local coord on U_j, z a local coord. on W, such that $\phi_W \circ F \circ \phi_{U_j}^{-1}$ is of form $z_j \mapsto z = z_j^{m_j}$ $m_j = \text{mult.}_{u_j}(F)$

Applying our knowledge from previous example, the element of S_d corresponding to a loop of winding # 1 around b in W is a product of cycles of size $m_1, \dots, m_k$. (Note $\sum_{j=1}^{k} m_j = d$)

If we are being careful about base points, then given base point in Y may not lie in W. So take path from base point y_0 to point y_1 in W, call it α, then apply small loop around b – call it β, then traverse back along α in opposite direction : α^{-1}.

Think of α as an identification of fiber of F over y_0 and fiber of F over y_1. If we view the fiber as labelling then different α may give different identifications of labelling. Thus elts. of S_d are only determined up to conjugation, but this preserves cycle type.

<u>Conclusion</u>: Given non-const., proper holom. map $F : X \to Y$, we obtain an integer $\underset{\text{degree}}{(d)}$, a discrete set $\underset{\text{(branch points)}}{B \subset Y}$, and a transitive gp. homom. $\underset{\text{(monodromy repn)}}{\rho : \pi_1(Y \setminus B) \to S_d}$ up to conjugacy.

<u>Thm</u>: Let Y be Riemann surface, B: discrete set in Y. $d \geq 1$ integer, $\rho : \pi_1(Y \setminus B) \to S_d$ transitive gp. hom., then there exists a pair (F, X) with $F : X \to Y$ proper holom. map of Riemann surfaces s.t. its monodromy repn is ρ. Such (F, X) are unique up to equivalence.

<u>Pf</u>. By the theory of covering spaces, if given a subgp. H of index d of $\pi_1(Y \setminus B)$ then we may form a cover $F_0 : X_0 \to Y \setminus B$ of degree d.

Pick an index $\in \{1, \dots, d\}$, say 1, consider $[\gamma] \in \pi_1(Y \setminus B)$ s.t. $\rho([\gamma])(1) = 1$