If we are being careful about base points, then a given base point in $Y$ may not lie in $W$. So take path from basepoint $y_0$ to point $y_1$ in $W$, call it $\alpha$, then apply small loop around $b$—call it $\beta$, then traverse back along $\alpha$ in opposite direction $= \alpha^{-1}$.

Think of $\alpha$ as an identification of fiber of $F$ over $y_0$ and fiber of $F$ over $y_1$. If we view the fiber as labelling then different $\alpha$ may give different identifications of labelling. Thus elts. of $S_d$ are only determined up to conjugation, but this preserves cycle type.

Conclusion: Given non-constant, proper holomorphic map $F : X \to Y$,
we obtain an integer $(d)$, a discrete set $B \subset Y$, and a (branch points)
transitive gp. homom. $\rho : \pi_1(Y \setminus B) \to S_d$, up to conjugacy.

( Monodromy repn)

**Thm**: Let $Y$ be a Riemann surface, $B$ a discrete set in $Y$.

If $d \geq 1$, integer, $\rho : \pi_1(Y \setminus B) \to S_d$, transitive gp. hom., then there exist a pair $(F, X)$ with $F : X \to Y$ a proper holomorphic map of Riemann surfaces s.t. its monodromy repn is $\rho$. Such $(F, X)$ are unique up to equivalence.

If $H$ By the theory of covering spaces, if give a subgroup of index $d$ of $\pi_1(Y \setminus B)$
then we may form a cover $F : X \to Y \setminus B$ of degree $d$.

Pick an index $\in \{1, \ldots, d\}$, say 1, consider $[y] \in \pi_1(Y \setminus B)$ s.t. $\rho([y])(1) = 1$
That is, \( \pi \) maps to a permutation fixing 1. These \( \pi \) form a subgroup \( \Pi \) of index \( d \). Take corresponding cover.

This realizes the monodromy rep\( \pi \) by our earlier abstract description.

Initially \( X_0 \) is just connected topological space. But since \( Y \), and hence \( Y \setminus B \), are R.S., then make charts for \( X_0 \) via composing covering map with charts for \( Y \setminus B \). Requiring that \( F_0 \) is holomorphic with respect to R.S. structure on \( X_0 \) specifies it uniquely.

Have: \( F_0 : X_0 \rightarrow Y \setminus B \) \( \text{Word} = F : X \rightarrow Y \).

Need to explain how to fill in pts. of \( F^{-1}(B) \) over branch points \( B \).

Pick \( b \in B \), small disk around \( b \), \( Y \) : boundary so that \( D_b \).

\( \Gamma(Y) \) defines cong class in \( \pi_1(Y \setminus B) \). Apply monodromy rep\( \pi \), then

\[ \rho(\pi) \text{ = permutation of cycle type } m_1, \ldots, m_k \text{ s.t. } \sum m_j = d. \]

Connected components of \( F^{-1}_0(D_b \setminus b) \) correspond to cycles.

Pick one, \( z \). Then \( z \) is cover of \( D_b \setminus b \) of degree \( m_j \).

With generator of \( \pi_1(D_b \setminus b) \) mapping to an \( m_j \)-cycle.

\[ \Rightarrow z \cong D^k : \text{disk in } C \text{ with map } z \rightarrow z^{m_j} \]

Taking us from \( z \rightarrow D_b \rightarrow D^k \).

Consider \( X = X_0 \sqcup / \theta \) where \( D \) is non-punctured disk in \( C \).
So points of $\mathbb{C} \times X_0$ are identified with pts. of $D^* \subset D$ via
$\phi$, and unique point not in $X_0$ is $\phi_0 \in D$.

Know $X_0 \sqcup D/\phi$ is R.S. provided we can show it is Hausdorff.

i.e. given $a, b \in X$, want disjoint open sets $U, V$ containing $a, b$ respectively.

if $a, b \in X_0$, done since $X_0$ is R.S. so in particular Hausdorff.

Only difficulty: $a \in X_0, b = \phi_0 \in D$. But $F_0 : X_0 \to Y \setminus B$ maps
$a \mapsto F_0(a) \neq b$ so $\exists$ open nbhd $N_u$ of $F_0(a)$ in $Y \setminus B$
which is disjoint from small open nbhd. $N_v$ of $b$.

$\Rightarrow F_0^{-1}(N_u)$ and $\phi_0 \cup \phi_0^{-1}(F_0^{-1}(N_v))$ are disjoint in $X$.

[In general can always try construction $X \sqcup D/\phi$ where $D^* \subset \subset X$

When will result be Hausdorff? If and only if

$\phi$ extends to holomorphic map $\tilde{\phi}$ from $X$ to $D$.

Picture:

- We don't want closure of $\mathbb{C}$ in $X$ to fill middle.
- (i.e. if point we were trying to "fill" already existed in $X$)

See p. 66 of Miranda

"Plugging holes in Riemann surfaces"
The fact that $F_0$ extends to a holomorphic map $F : X \to Y \setminus \bigcup_{b \in B} B_b$ is clear, since locally it is an $n$th power map, which has unique extension at $b$.

This isn't the end; we need to repeat this for each cycle in monodromy rep $\rho$ of $[\Sigma]$ for $b$, then repeat for each $b \in B$.

Then done, upon checking the result is proper. For example if $Y$ is compact, then $X$ compact since it is expressible as union of finitely many compact sets: $F_0^{-1}(Y \setminus D_b)$, $D_b$ : open nbhd. of $b \in B$

so $Y \setminus D_b$ compact

and closures of open discs from hole chalks for cycles in $\rho [\partial D_b]$.

Next time: revisit compactifying algebraic curves (e.g. hyperelliptic)

with new understanding of monodromy / plugging holes.

Wont assume it is smooth. Just polynomial in two variables in $\mathbb{C}^2$ irreducible, not linear of form $z - z_0$.

Facts: $X = \bigcap_{\ell \in \mathbb{C}^2} P(\ell z) = 0$ then $P$ as above

- $X$ is connected.
- There are only finitely many points $(z, w)$ where $P, \frac{\partial P}{\partial w}$ both vanish.

$J$: singular set $= \bigcap (z, w) \in X | \frac{\partial P}{\partial z} = \frac{\partial P}{\partial w} = 0$ (finite by fact above)

$X \setminus S$ is Riemann surface. Want to compactify this.