Last time, we were studying singular algebraic curves \( P(\nu, \varphi) = 0 \).

\[
\Sigma^+ := \pi_2^{-1}(\pi_2(\Sigma \cup F)) \quad \Sigma: \text{singular points}
\]

\[ F: \text{zeros of highest power of } \nu \text{ in } \mathbb{P}(\nu, \varphi) \]

\[ \pi_2 : X \setminus \Sigma^+ \longrightarrow S^2 \setminus E \quad \text{where} \]

\[ E = \pi_2(\Sigma) \cup F \cup \{0\} \]

proper holomorphic map \( \sim \) monodromy rep \( \rho : \pi_1(S^2 \setminus (E \cup B)) \longrightarrow S_4 \)

\[ B: \text{branch pts} \]

Example: \( W^2 - z^2(1-z) = 0 \)

Single \# singularity at \((0,0)\).

branch point at \( z = 1 \) (mult. 2)

\[ B \cup E = \{0, 1, 0, 3\} \quad \pi_2 : \text{deg. 2} \]

Analyse monodromy rep \( \rho : \pi_1(S^2 \setminus \{0, 1, 0, 3\}) \longrightarrow S_4 \)

If \( U : \text{nbhd of } 0 \text{ in } S^2 \setminus \{0, 1, 0, 3\} \), \( \pi_2^{-1}(U) \) is disjoint union of two punctured open sets

thus monodromy rep \( \rho \) maps

\[ [Y], Y \text{ loop of winding \#1 about } \{0\} \]

\[ \rho(1)(2): \text{identity permutation} \]

So to form \( X^* \), attach 2 disks to \( X \setminus \Sigma^+ \), one for each component.

Also attach 1 disk for branch pt \( 1 \), 1 disk for \( \{0, 3\} \), each of mult. 2.

What is genus of resulting R.S.? Hurwitz formula: \( \pi_2^\# : X^* \longrightarrow S^2 \quad \text{deg 2} \).
Hermite: \[ 2g(X^*) - 2 = \text{deg}(\pi_2) (2g(S^2) - 2) + \sum_{p \in X} \text{mult}_p (\pi_2)_1 - 1 \]

\[ = 2 \cdot (-2) + 2 \text{ (from branch pts. with mult. 2) } \]

\[ \Rightarrow g(X^*) = 0 \text{ so } X^* \text{ isomorphic to Riemann sphere.} \]

**Proposition:** \( \bar{X} \): homogenized zero locus for \( P(2;w) \), a compact set in \( \mathbb{P}^2(\mathbb{C}) \).

Then natural inclusion \( X \setminus \Sigma^+ \hookrightarrow \bar{X} \) extends to a \( (z,w) \mapsto [\bar{z}:w:1] \)

holomorphic map \( X^* \to \mathbb{P}^2(\mathbb{C}) \) mapping onto \( \bar{X} \).

Need to ensure that the inclusion extends to holomorphic map at centers of glued disks making \( X^* \) from \( X \setminus \Sigma^+ \).

Recall that when we paste in disk \( D \), do this by considering covering map from punctured nbhd \( U \) of \( (z_0,w_0) \) to punctured nbhd \( V \) of \( z_0 \) given by \( \pi_2 \). Any covering space is itself isomorphic to a disk \( B \setminus D \) with covering map: \( y \mapsto y^m \) for some \( m \).

Identifying \( U \setminus (z_0,w_0) \) with \( B \setminus D \) under homeomorphism \( \phi \), making the following diagram commute:

\[
\begin{array}{ccc}
U \setminus (z_0,w_0) & \xrightarrow{\phi} & B \setminus D \\
(z,w) & \xrightarrow{\pi_2} & D \setminus z_0 \\
\end{array}
\]

where \( w(y) \) given by composition of \( y \mapsto y^m \) and function guaranteed by implicit function theorem.
Last time we proved \( w(y) \) may be extended to a meromorphic function on the whole disk, thus we have
\[
y \mapsto [y^n; w(y); 1] \text{ is a meromorphic map from disk } B \to \mathbb{P}^2(\mathbb{C})\]
But on the other hand, on punctured disk \( B \setminus \{0\} \)
it equals \( y \mapsto [y^{m+n}; y^n w(y); y^n] \) where \( n \) is order of pole at \( y=0 \), so
\[
y^n w(y) \text{ is non-vanishing at } 0.
\]
So we've constructed the "normalization" of \( \overline{X} \).

The example is a general phenomenon - the normalization of singular curves produces compact R.S. with different genus than that of its non-singular counterparts. Elliptic curves: genus 1, singular cubics: genus 0.

Try it for \( w^2 = z^3 \).

Algebraic curves - see Brieskorn. (available electronically from library)
many interesting topics: Puiseux expansions
(factoring in \( \mathbb{P}(\mathbb{C}[w]) \) using fractional powers
algorithms based on Newton polygon)

nice discussion of alternate topologies,
germs of functions.

moduli problem - classify all curves of given genus up to isomorphism.

E.g. elliptic curves: \( J \)-invariant to \( \mathbb{C} \), generalizations to higher genus by Mumford.
A little bit about Puiseux expansions:

For simplicity, suppose \((0,0)\) is our singular pt, so that

\[ P(z,w) \text{ has power series expansion } = \sum_{m,n} p_{m,n} z^m w^n, \]

To construct "normalization"

"gluing in disks according to cycle type in monodromy."

So if \(\pi_1\) has degree \(d_1\) as map on \(X \setminus \mathbb{A}^1\), then size of cycles adds to \(d_1\)

Easiest case: monodromy trivial, as in nodal example,

then analyzing function \(z \to P(z, f_i(z)) \) for some holomorphic functions \(f_i\)

and can factor the corresponding \(P\) as:

\[ P(z,w) = (w - f_1(z))(w - f_2(z)) \cdots (w - f_{d_1}(z)) Q(z,w) \]

with \(Q \neq 0\) at \((0,0)\).

What if we have cycle of length \(a\)?

This corresponds to local coord map \(z = y^a\)

so analyze \(P(y^a, w)\), and look for factors of the form \(w - f(y^a)\).

If \(w - f(y)\) is a factor, so is \(w - f(sby)\)

\((s = e^{2\pi i/a}, b = 0, \ldots, a-1)\)

(since \(f\) is made from

implies function \(g\) and composite \(y = y^a<\)

\[ P(y, w) = \left( w - f(s) \right) \left( w - f(sy) \right) \cdots \left( w - f_{a-1}(y) \right) Q(y,w) \]

with \(z = y^a\).
Substituting, we get \( P(z, w) = (w - f_0(z^{1/a})) \cdots (w - f_{a-1}(z^{1/a})) \)

i.e. adjoin formal variables \( z^{1/a} \) for suitable \( a \), with \( Q(z^{1/a}, w) \)

to obtain factorization \( \sum a_i = d \).
\( W^2 = z^2(1-z) \)

W = tz hits line \( z = t - 1 \) at \( W = -t \)

and hits cubic at \( (t^2, t^2) = z^2(1-z) \)

\( z = 1 - t^2 \)

\( W = t - t^3 \)

Gives map \( t \mapsto (1-t^2, t-t^3) \)

sends \( t = \pm 1 \) to the singular pt. \( (0,0) \).

\[ \phi : \mathbb{C} \to \mathbb{R} \times \text{non-singular curve.} \]