Last time, we were studying singular algebraic curves $P(w,z)=0$.

$$\Sigma^+ := \pi_z^{-1}(\pi_z(\Sigma) \cup F)$$

Σ : singular points

F : zeros of highest power of w in $P(w,z)$ in z

$\pi_z : X \setminus \Sigma^+ \longrightarrow S^2 \setminus E$ where

$\pi_z : X \longrightarrow \mathbb{C}$
$(z,w) \longmapsto z$

$$E = \pi_z(\Sigma) \cup F \cup \{\infty\}.$$

proper holomorphic map. $\rightsquigarrow$ monodromy rep'n

$$\rho : \pi_1(S^2 \setminus (E \cup B)) \longrightarrow S_d$$

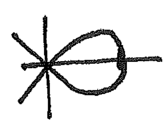
B : branch pts

$\Big\}$

$\downarrow$

compact R.S. X^*.

Example : $w^2 - z^2(1-z) = 0$



Single singularity at $(0,0)$.

branch point at $z=1$ (mult. 2)

$B \cup E = \{0,1,\infty\}$. π_z : deg. 2

Analyze monodromy rep'n $\rho : \pi_1(S^2 \setminus \{0,1,\infty\}) \longrightarrow S_2$

If $\mathcal{U}$: nbhd of 0 in $S^2 \setminus \{0,1,\infty\}$, $\pi_z^{-1}(\mathcal{U})$ is disjoint union of two punctured open sets one for each of $\pm 1 = w$

thus monodromy rep'n maps

$[\gamma]$, γ loop of winding #1 about $\{0\}$

to $(1)(2)$: identity permutation.

So to form X^*, attach 2 disks to $X \setminus \Sigma^+$, one for each component.

Also attach 1 disk for branch pt 1, 1 disk for $\{\infty\}$, each of mult. 2.

What is genus of resulting R.S.? Hurwitz formula: $\pi_z : X^* \longrightarrow S^2$ deg 2.

Hurwitz: $2g(X^*) - 2 = \deg(\pi_z)\,(2g(S^2)-2) + \sum_{p \in X} \mathrm{mult}_p(\pi_z) - 1$

$= 2 \cdot (-2) + 2$ (from $\{\infty, 1\}$ branch pts. with mult. 2)

$\Rightarrow g(X^*) = 0$ so X^* isomorphic to Riemann sphere.

Proposition: $\overline{X}$: homogenized zero locus for $P(z,w)$, a compact set in $\mathbb{P}^2(\mathbb{C})$.

Then natural inclusion $X \setminus \Sigma^+ \hookrightarrow \overline{X}$, $(z,w) \mapsto [z:w:1]$ extends to a holomorphic map $X^* \to \mathbb{P}^2(\mathbb{C})$ mapping onto $\overline{X}$.

Need to ensure that the inclusion extends to holomorphic map at centers of glued disks making X^* from $X \setminus \Sigma^+$.

Recall that when we paste in disk D, do this by considering covering map from punctured nbhd $\mathcal{U}$ of (z_0, w_0) to punctured nbhd V of z_0 given by π_z. Any covering space is itself isomorphic to a disk $B - \{0\}$ with covering map: $B \setminus \{0\} \to D \setminus \{z_0\}$, $y \mapsto y^m$ for some m.

Identifying $\mathcal{U} \setminus (z_0, w_0)$ with $B \setminus \{0\}$ under homeom. ϕ, making the following diagram commute:

$$\begin{array}{ccc} \mathcal{U} \setminus \{(z_0,w_0)\} & \xrightarrow{\phi} & B \setminus \{0\} \\ (z,w) & & y \\ & \searrow^{\pi_z} \quad \swarrow & \\ & D \setminus \{z_0\} & \\ & z \qquad y^m & \end{array}$$

$\phi(y) = (y^m, w(y))$

where $w(y)$ given by composition of $y \to y^m$ and function guaranteed by implicit function theorem.

Last time we proved $w(y)$ may be extended to a meromorphic function on the whole disk, thus we have

$$y \longmapsto [\, y^m : w(y) : 1 \,]$$ is a meromorphic map from disk B to $\mathbb{P}^2(\mathbb{C})$

But on the other hand, on punctured disk $B \setminus \{0\}$

it equals $y \longmapsto [\, y^{m+n} : y^n w(y) : y^n \,]$ where n : order of pole at $y=0$ so $y^n w(y)$ is non-vanishing at 0.

and this extends to holom. map on B.

So we've constructed the "normalization" of $\overline{X}$.

The example is a general phenomenon – the normalization of singular curves produces compact R.S. with different genus than that of its non-singular counterparts. Elliptic curves: genus 1, singular cubics: genus 0.

Try it for $w^2 = z^3$.

Algebraic curves – see Brieskorn. (available electronically from library)

many interesting topics: Puiseux expansions

(factoring $P(z,w)$ using fractional powers of z

algorithms based on Newton polygon)

nice discussion of alternate topologies, germs of functions.

Moduli problem – classify all curves of given genus up to isomorphism.

e.g. elliptic curves : j-invariant to $\mathbb{C}$, generalizations to higher genus by Mumford.

A little bit about Puiseux expansions:

for simplicity, suppose $(0,0)$ is our singular pt. so that

$P(z,w)$ has power series expansion $= \sum_{\substack{m,n \\ m+n\geq 2}} p_{m,n} z^m w^n$

To construct "normalization"

gluing in disks according to cycle type in monodromy.

so if π_z has degree d_1 as map on $X \setminus \Sigma^+$, then size of cycles adds to d_1

Easiest case: monodromy trivial, as in node example,

then ~~[illegible]~~ analyzing function $z \mapsto P(z, f_i(z))$ for some holomorphic functions f_i $i=1,\dots,d_1$

and can factor the corresponding P as:

$$P(z,w) = (w - f_1(z))(w - f_2(z)) \cdots (w - f_{d_1}(z))\, Q(z,w)$$

with $Q \neq 0$ at $(0,0)$.

What if we have cycle of length a?

This corresponds to local coord map $z = y^a$

so analyze $P(y^a, w)$, and look for factors of the form $w - f(y)$.

If $w - f(y)$ is a factor, so is $w - f(\xi^b y)$

$\xi = e^{2\pi i/a}$, $b = 0,\dots,a-1$.

(since f is made from implicit function g and composite $y \mapsto y^a$.)

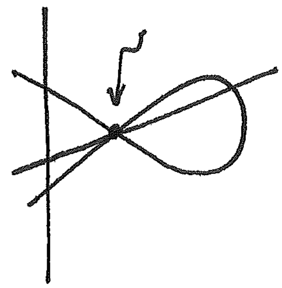
$$P(y^a, w) = (w - \underbrace{f(y)}_{\substack{\shortparallel \\ f_0(y)}})(w - \underbrace{f(\xi y)}_{\substack{\shortparallel \\ f_1(y)}}) \cdots (w - f_{a-1}(y))\, Q(y,w)$$

with $z = y^a$.

Substituting, we get $P(z,w) = (w - f_0(z^{1/a})) \cdots (w - f_{a-1}(z^{1/a}))$
$Q(z^{1/a}, w)$

i.e. adjoin formal variables z^{1/a_i} for suitable a_i with $\sum a_i = d_1$
to obtain factorization

$$w^2 = z^2(1-z)$$



$z=-1$

picture of real locus, but we want to take $t \in \mathbb{C}$ of course.

$w = tz$ hits line $z = -1$

at $w = -t$

and hits cubic at $(t^2 z^2) = z^2(1-z)$

$$z = 1 - t^2$$
$$w = t - t^3$$

Gives map $t \longmapsto (1-t^2, t-t^3)$

sends $t = \pm 1$ to the singular pt. $(0,0)$.

$$\phi : \underbrace{\mathbb{C}}_{\text{non-singular curve.}} \longrightarrow X$$