Differential forms – use them even in one-variable calculus, when we write $\int_a^b \underbrace{f(z)\,dz}_{\text{1-form}}$. Think of dz as keeping track of coordinate z with set of rules for how to manipulate infinitesimal "chunk of area" under transformations of z.

Abstract point of view: Define holomorphic 1-form on open set $U \subseteq \mathbb{C}$ to be an expression $\omega := f(z)\,dz$ with f: holomorphic on U. (w.r.t. coordinate z)

Then say that ω_1 transforms to ω_2 if there exists holomorphic map $T: U_2 \to U_1$, $w \mapsto z$ such that (setting $\omega_1 = f(z)dz$, $\omega_2 = g(w)\,dw$)

$$g(w) = f(T(w))\,T'(w) \qquad \left(\text{or equivalently } \omega_2 = f(T(w))\,\underbrace{d\,T(w)}_{T'(w)\,dw}\right)$$

Thus holomorphic 1-form on Riemann surface is compatible collection of 1-forms $\{\omega_\phi\}$ $\qquad \phi: U \to V \subseteq \mathbb{C}$ is chart.

Compatible: ω_{ϕ_1} transforms to ω_{ϕ_2} under $T = \phi_1 \circ \phi_2^{-1}$.

Reconsider (real) differential forms in coordinate-free way.

one variable: small displacement $\Delta x_i = x_{i+1} - x_i \longmapsto \underbrace{f(x_i)}_{\text{constant of proportionality}}\Delta x_i$

multivariable: ω_{x_i}: linear transformation $\mathbb{R}^n \to \mathbb{R}$ taking Δx_i (now vector displacement) to $\omega_{x_i}(\Delta x_i)$

i.e. ω_{x_i} is linear functional on space of tangent vectors at x_i in $\mathbb{R}^n$
"cotangent vector" at x_i

1-form is a continuous assignment of cotangent vectors for each pt. $x_i \in \mathbb{R}^n$.

Lemma : f: smooth, real-valued function on nbhd $\mathcal{U}$ of origin in $\mathbb{R}^2$.

$\gamma_1 : (-\varepsilon_1, \varepsilon_1) \to \mathcal{U}$ smooth maps $\gamma_i(0) = 0$.

$\gamma_2 : (-\varepsilon_2, \varepsilon_2) \to \mathcal{U}$ $\varepsilon_1, \varepsilon_2 > 0$.

$\chi : \mathcal{U} \to V$ diffeomorphism to open set V of $\mathbb{R}^2$ with $\chi(0) = 0$.

$\tilde{\gamma}_i = \chi \circ \gamma_i$

$\tilde{f} = f \circ \chi^{-1}$

then (by chain rule) ① if $\left.\frac{\partial f}{\partial x_1}\right|_0 = \left.\frac{\partial f}{\partial x_2}\right|_0 = 0$, then $\left.\frac{\partial \tilde{f}}{\partial x_1}\right|_0 = \left.\frac{\partial \tilde{f}}{\partial x_2}\right|_0 = 0$

② if $\left.\frac{d\gamma_1}{dt}\right|_0 = \left.\frac{d\gamma_2}{dt}\right|_0$ then $\left.\frac{d\tilde{\gamma}_1}{dt}\right|_0 = \left.\frac{d\tilde{\gamma}_2}{dt}\right|_0$.

If S : 2-dim'l real smooth (i.e. C^∞) manifold, $p \in S$.

f smooth function on S, $\gamma_i : (-\varepsilon_i, \varepsilon_i) \to S$ smooth paths $i = 1, 2$.

$\gamma_i(0) = p$

Then lemma implies we may define:

f is "constant to first order" at p if the derivative of f (in local coords near p) vanishes at p.

(Key pt : lemma implies independent of chart.)

same fact used before to argue that order of holomorphic/merom. function on R.S. is well-defined.

Similarly, say γ_1, γ_2 are "equal to first order" if derivatives w.r.t. local chart at p are equal

Define : Tangent space TS_p of S at p is the set of equivalence classes of smooth maps $\gamma : (-\varepsilon, \varepsilon) \to S$ with $\gamma(0) = p$ where $\gamma_1 \sim \gamma_2$ if they are equal to first order at p.

Similarly, the cotangent space T^*S_p of S at p is the set of equivalence classes of smooth functions on open nbhd of $p \in S$, with $f_1 \sim f_2$ if $f_1 - f_2$ constant to first order at p.

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Given $f \in C^\infty(\mathcal{U})$, $p \in \mathcal{U}$, there exists corresp. elt. in T^*S_p, call it $[df]_p$. Given local coords x_1, x_2 for $\mathcal{U}$, these too are smooth functions, with corresp. elts $\underbrace{[dx_1]_p, [dx_2]_p}$.

might really be better to call charts ϕ_1, ϕ_2 with local coords x_1, x_2

so maps are to $d\phi_1, d\phi_2$ but this is not standard notation.

Writing $f = f(x_1, x_2)$

$$[df]_p = \frac{\partial f}{\partial x_1}\Big|_p [dx_1]_p + \frac{\partial f}{\partial x_2}\Big|_p [dx_2]_p$$

and clearly $[dx_1]_p, [dx_2]_p$ form a basis of T^*S_p.

(To see that $T^*S_p \simeq \mathrm{Hom}(TS_p, \mathbb{R})$, note we have a bilinear pairing which respects equivalence classes)

$$TS_p \times T^*S_p \longrightarrow \mathbb{R}$$
$$(\gamma, f) \longmapsto \frac{d}{dt}(f \circ \gamma)\Big|_{t=0}$$

Cotangent <u>bundle</u> : $T^*S := \bigcup_{p \in S} T^*S_p$.

smooth <u>1-form</u> : $\alpha : S \to T^*S$ with α smooth.

so have to check that in local coords, say (x_1, x_2) about p,

$\alpha = \alpha_1 \, dx_1 + \alpha_2 \, dx_2$ that $\alpha_1(x_1, x_2)$, $\alpha_2(x_1, x_2)$ are smooth.

As usual, this is independent of choice of chart (check this!) chain rule...

easiest way to define 1-form: Start with smooth function f on all of S.

Take $$df(p) = \left.\frac{\partial f}{\partial x_1}\right|_p dx_1 + \left.\frac{\partial f}{\partial x_2}\right|_p dx_2.$$

But not all smooth 1-forms arise in this way.*

Can also produce them from pullback construction: $F: S \to Q$ smooth map of C^∞-manifolds

define $dF_p : TS_p \longrightarrow TQ_{F(p)}$

$dF_p^* : T^*Q_{F(p)} \longrightarrow T^*S_p$

given α: smooth 1 form on Q, make $F^*(\alpha)$ defined by

$$F^*(\alpha)(p) = dF_p^*(\alpha) = \alpha \circ F$$

Let $d : \underbrace{\Omega^0_S}_{\text{smooth functions}} \longrightarrow \underbrace{\Omega^1_S}_{\text{smooth 1-forms}}$ as above. It satisfies $d(fg) = f\,dg + g\,df$

$d(F^*f) = F^*(df)$ $F: S \to Q$ smooth.

Finally define integration for smooth path $\gamma : [0,1] \to S$ on 1-form α:

$$\int_\gamma \alpha = \sum_i \int_{\gamma^{(i)}} \alpha_1(\gamma_1^{(i)}(t), \gamma_2^{(i)}(t)) \frac{d\gamma_1^{(i)}}{dt} + \alpha_2(\gamma_1^{(i)}(t), \gamma_2^{(i)}(t)) \frac{d\gamma_2^{(i)}}{dt}$$

$\gamma^{(i)}$ are pieces of original curve γ lying in individual coordinate charts.

(of course, $\gamma^{(i)}$ partition γ.)

i.e. $[t_i, t_{i+1}]$ defining $\gamma^{(i)}$ partition $[0,1]$.

(*) Write $\alpha = \alpha_1\, dx_1 + \alpha_2\, dx_2$

α_1, α_2 smooth functions.

Equality of mixed partials $\Rightarrow$ if $\alpha = df$ then $\dfrac{\partial \alpha_1}{\partial x_2} = \dfrac{\partial \alpha_2}{\partial x_1}$. Necessary condition. Also sufficient.