Differential forms – use them even in one-variable calculus, when we write \( \int_a^b f(z) \, dz \). Think of \( dz \) as keeping track of coordinate \( z \) with set of rules for how to manipulate infinitesimal "chunk of area" under transformations of \( z \).

**Abstract point of view:** Define holomorphic 1-form on open set \( U \subset \mathbb{C} \) to be an expression \( \omega \Rightarrow f(z) \, dz \) with \( f \) holomorphic on \( U \).

(W.r.t. coordinate \( z \))

Then say that \( \omega_1 \) transforms to \( \omega_2 \) if there exists

holomorphic map \( T : U_2 \rightarrow U_1 \) such that \( \omega_2 = g(w) \, dw \quad (w \rightarrow z) \)

\[ g(w) = f(T(w)) \frac{dT}{dw} \quad \text{(or equivalently) \quad \omega_2 = f(T(w)) \frac{d(T(w))}{dw} \omega_1} \]

Thus holomorphic 1-form on Riemann surface is compatible collection of 1-forms \( \{ \omega_\phi \} \phi : U \rightarrow V \subset \mathbb{C} \) is chart.

Compatible: \( \omega_\phi \) transforms to \( \omega_{\phi_2} \) under \( T = \phi_1 \circ \phi_2^{-1} \).

Reconsider (real) differential forms in coordinate-free way.

one variable: small displacement \( \Delta x_i = x_{i+1} - x_i \) \( \rightarrow \frac{f(x_i)}{\Delta x_i} \) constant of proportionality

multivariable: \( \omega_{x_i} \): linear transformation \( \mathbb{R}^n \rightarrow \mathbb{R} \) taking \( \Delta x_i \) (new vector displacement) to \( \omega_{x_i}(\Delta x_i) \)

i.e. \( \omega_{x_i} \) is linear functional on space of tangent vectors at \( x_i \) in \( \mathbb{R}^n \) "cotangent vector" at \( x_i \)

1-form is a continuous assignment of cotangent vectors for each pt. \( x_i \in \mathbb{R}^n \).
Lemma: \( f \): smooth, real-valued function on neighborhood \( U \) of origin in \( \mathbb{R}^2 \).

\[ \gamma_1: (-\varepsilon_1, \varepsilon_1) \to U \text{ smooth maps \ } \gamma_i(0) = 0. \]
\[ \gamma_2: (-\varepsilon_2, \varepsilon_2) \to U \text{ \ } \varepsilon_1, \varepsilon_2 > 0. \]

\[ \chi: U \to V \text{ diffeomorphism to open set } V \text{ of } \mathbb{R}^2 \text{ with } \chi(0) = 0. \]

Then (by chain rule) if \( \frac{\partial f}{\partial x_1}(0) = \frac{\partial f}{\partial x_2}(0) = 0 \), then \( \frac{d\gamma_i}{d\tau}(0) = 0 \).

(2) if \( \frac{d\gamma_1}{dt}(0) = \frac{d\gamma_2}{dt}(0) \), then \( \frac{d\gamma_i}{dt}(0) = \frac{d\gamma_i}{dt}(0) \).

If \( S \): 2-dim'l real smooth \( (C^m) \) manifold, \( p \in S \).

\( f \): smooth function on \( S \), \( \gamma_i: (-\varepsilon_1, \varepsilon_1) \to S \text{ smooth paths } i=1,2. \)

\( \gamma_i(0) = p \)

Then lemma implies we may define:

\( f \) is "constant to first order" at \( p \) if the derivative of \( f \) (in local coords) near \( p \) vanishes at \( p \).

(Keypoint: lemma implies independent of chart.)

Some fact used before to argue that order of holomorphic/meromorphic function on R.S. is well-defined.

Similarly, say \( \gamma_1, \gamma_2 \) are equal to first order if derivatives w.r.t. local chart at \( p \) one equal.

Define: Tangent space \( T_S p \) of \( S \) at \( p \) is the set of equivalence classes of smooth maps \( \gamma: (-\varepsilon, \varepsilon) \to S \text{ with } \gamma(0) = p \)

where \( \gamma_1 \sim \gamma_2 \) if they are equal to first order at \( p \).
Similarly, the cotangent space $T^*S_p$ of $S$ at $p$ is the set of equivalence classes of smooth functions on open nbhd of $p \in S$, write $f_1 \sim f_2$ if $f_1 - f_2$ constant to first order at $p$.

Given $f \in C^0(U)$, $p \in U$, there exists corr. elt. in $T^*S_p$, call it $[df]_p$. Given local coords $x_1, x_2$ for $U$, these too are smooth functions, with corr. elt $[dx_1]_p, [dx_2]_p$.

Writing $f = f(x_1, x_2)$

$$[df]_p = \frac{df}{dx_1} [dx_1]_p + \frac{df}{dx_2} [dx_2]_p$$

and clearly $[dx_1]_p, [dx_2]_p$ form a basis of $T^*S_p$.

To see that $T^*S_p \cong \text{Hom}(T_pS, \mathbb{R})$, note we have a bilinear pairing $T_pS \times T^*S \to \mathbb{R}$

$$(x, f) \mapsto \left. \frac{d}{dt} (f \circ x) \right|_{t=0}$$

Cotangent bundle: $T^*S := \coprod_{p \in S} T^*S_p$.

Smooth 1-form: $\alpha : S \to T^*S$ with $\alpha$ smooth.

So have to check that in local coords, say $(x_1, x_2)$ about $p_1$

$$\alpha = \alpha_1 dx_1 + \alpha_2 dx_2$$

that $\alpha_1(x_1, x_2), \alpha_2(x_1, x_2)$ are smooth.

As usual, this is independent of choice of chart (check this!) clean rule...
easiest way to define 1-form: Start with smooth function $f$ on all of $S$.

Take \( d\delta f(p) = \left. \frac{df}{dx_1} \right|_p dx_1 + \left. \frac{df}{dx_2} \right|_p dx_2 \).

But not all smooth 1-forms arise in this way.*

Can also produce them from pullback construction: \( F: S \to Q \) smooth map of \( C^\infty \)-manifolds

\[ dF_p: T_S p \to T_Q f(p) \]
\[ dF^*: T^* Q f(p) \to T^* S_p \]

given \( \alpha \) smooth 1-form on $Q$, make \( F^*(\alpha) \) defined by

\[ F^*(\alpha)(p) = dF^* (\alpha \circ f) = \alpha \circ F \]

Let \( d: \Omega^1_S \to \Omega^1_S \) as above. It satisfies \( d(fg) = f \, dg + g \, df \).

\[ d(F^* f) = F^* (df) \]

Finally define integration for smooth path

\( \gamma: [0,1] \to S \) on 1-form $d$:

\[ \int d = \sum_i \int_{\gamma^{(i)}} \alpha_1 (\gamma_1^{(i)}(t), \gamma_2^{(i)}(t)) \frac{d\gamma_1^{(i)}}{dt} + \alpha_2 (\gamma_1^{(i)}(t), \gamma_2^{(i)}(t)) \frac{d\gamma_2^{(i)}}{dt} \]

\( \gamma^{(i)} \) are pieces of original curve \( \gamma \) lying in individual coordinate charts.

(If, of course, \( \gamma^{(i)} \) partition $\gamma$.)

i.e. \([0,1] \to [0,1] \) defining \( \gamma^{(i)} \) partition $[0,1]$.

(+) Write \( \alpha = \alpha_1 \, dx_1 + \alpha_2 \, dx_2 \)

\( \alpha_1, \alpha_2 \) smooth functions.

Equality of mixed partials \( \Rightarrow \) if \( \alpha = df \) then \( \frac{\partial \alpha_1}{\partial x_2} = \frac{\partial \alpha_2}{\partial x_1} \). Necessary condition.

Also sufficient.