Last time, defined smooth 1-form α on surface S

as map $\alpha : S \to T^*S = \bigcup_{p \in S} T^*S_p$ (cotangent bundle)

s.t. $\alpha = \alpha_1 \, dx_1 + \alpha_2 \, dx_2$ in local coords at p_0
($\forall \, p_0 \in S$) with α_i smooth in nbhd of p_0.

natural source: f : smooth function on $S \overset{d}{\longmapsto} df = \dfrac{\partial f}{\partial x_1} dx_1 + \dfrac{\partial f}{\partial x_2} dx_2$.

~~Thus~~ So have map $d : \underbrace{\Omega^0_S}_{\substack{\text{smooth functions} \\ \text{(0-forms)}}} \longrightarrow \underbrace{\Omega^1_S}_{\substack{\text{smooth} \\ \text{1-forms}}}$

Not surjective. claimed $\alpha = \alpha_1 \, dx_1 + \alpha_2 \, dx_2$ is equal to df for some smooth f on S ("exact" form)

if and only if $\dfrac{\partial \alpha_1}{\partial x_2} - \dfrac{\partial \alpha_2}{\partial x_1} = 0$. (**)

necessary: equality of mixed partials of f.

sufficient: Stokes' theorem — define $f_1(x_1, x_2) = \displaystyle\int_0^{x_2} \alpha_2(0,t)\, dt + \int_0^{x_1} \alpha_1(t, x_2)\, dt$

and $f_2(x_1,x_2) = \displaystyle\int_0^{x_1} \alpha_1(t,0)\, dt + \int_0^{x_2} \alpha_2(x_1, t)\, dt$

thus by FTC $\partial f_i / \partial x_i = \alpha_i(x_1, x_2)$. Done if we can show when (**) holds, $f_1(x_1,x_2) = f_2(x_1,x_2)$

But by Stokes' theorem: (Green's thm.) applied to rectangle R with vertices $(0,0), (x_1,0), (0,x_2), (x_1,x_2)$

$$\int_{\partial R} \alpha = f_1(x_1,x_2) - f_2(x_1,x_2) = \int_R \underbrace{\left(\frac{\partial \alpha_1}{\partial x_2} - \frac{\partial \alpha_2}{\partial x_1} \right)}_{= 0} dx_1 \, dx_2$$

so indeed $f_1 = f_2$. //

Why is this natural? Answer: Study 2-forms

Two <u>forms</u> : E : vector space over $\mathbb{R}$ (imagine TS_p - 2 dim'l v.s.)

Set $\Lambda^2 E^*$: bilinear maps $B : E \times E \to \mathbb{R}$ which

satisfy $B(e,f) = -B(f,e)$ "skew-symmetry"

Given pair of linear functions $\alpha, \beta \in E^* = \mathrm{Hom}(E, \mathbb{R})$, define

wedge product $\wedge : E^* \times E^* \to \Lambda^2 E^*$

$$\alpha, \beta \longmapsto (\alpha \wedge \beta)(e,f) := \alpha(e)\beta(f) - \beta(e)\alpha(f)$$

Note $\alpha \wedge \beta = -\beta \wedge \alpha$. <u>claim</u> : if α_1, α_2 are basis for E^*, then $\alpha_1 \wedge \alpha_2$ is basis (1-dim'l) for $\Lambda^2 E^*$ (*)

(If we return to our intuition from infinitesimals. - 2 forms are supposed to be objects to be integrated over 2-dim'l sets.

1-dimension : paths $[0,1] \to \mathbb{R}^n$, now maps $[0,1]^2 \to \mathbb{R}^n$ which we chop into infinitesimal squares of dimension $\Delta_1 x \wedge \Delta_2 x$ from base point x in $\mathbb{R}^n$. 2-form associates number $\omega_x(\Delta_1 x \wedge \Delta_2 x)$ to this.)

skew symmetry is recording orientation of these two vectors.

For $TS_p = E$, with x_1, x_2 local coords at p, then T^*S_p has basis $\{dx_1, dx_2\}$ and $\Lambda^2 T^* S_p$ has 1-dim'l basis $dx_1 \wedge dx_2$.

(sometimes write just $dx_1 dx_2$, dropping "$\wedge$")

Then 2-form $\rho : S \to \bigcup_{p \in S} \Lambda^2 T^* S_p$ so that when we express

$\rho = F(x_1, x_2)\, dx_1\, dx_2$ then F is smooth function on S.

If we change coordinates and follow through with definitions, y_1, y_2 second set of local coords:

$$\rho = F(x_1(y_1,y_2), x_2(y_1,y_2)) \underbrace{\left[\frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial y_1} \right]}_{\text{Jacobian}} dy_1\, dy_2$$

Jacobian - det. of matrix of first partials for proof of claim (*): bilinear forms are associated to a matrix, once we've chosen basis for the underlying vector space. Write

$B(v,w) = v^T M w$. If B is symmetric, then $M = M^T$

if B is skew-symmetric then $-M = M^T$

Strong requirement in 2×2 case : $M = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \Rightarrow B(v,w) = -a \cdot (v_2 w_1 - v_1 w_2)$

(one-dim'l space spanned by wedge of one forms $\alpha_1 \wedge \alpha_2$, once we appropriately translate to our situation.)

The skew-symmetry also implies that if α is 1-form, then $\alpha \wedge \alpha = 0$.

Back to Jacobian : can also think of this as result of pullback map with $F: S \to Q$ smooth map of surfaces, giving rise to map

$\Lambda^2 T^* Q_{F(p)} \to \Lambda^2 T^* S_p$ $\forall\ p \in S$.

(wedge of pullbacks of 1-forms)

Still haven't explained appearance of term in Stokes' / Green's thm.

Lemma : $\exists!$ $\mathbb{R}$-linear map $d: \underbrace{\Omega^1_S}_{\text{1-forms}} \to \underbrace{\Omega^2_S}_{\text{2-forms}}$ s.t.

(1) if α, β 1-forms on open set $U \subseteq S$, then $d\alpha = d\beta$ on U

(2) if f : smooth function on S, $d(df) = 0$.

(3) if f, α as above, $d(f\alpha) = df \wedge \alpha + f\, d\alpha$

pf : Such a map is unique since, by (1), we can calculate $d\alpha$ using local coordinates. If in nbhd U, $\alpha = \alpha_1 dx_1 + \alpha_2 dx_2$ then (2)+(3) imply :

$$d(\alpha_1 dx_1 + \alpha_2 dx_2) = d\alpha_1 \wedge dx_1 + d\alpha_2 \wedge dx_2$$

since $d(dx_i) = 0$ by (2). α_1, α_2 functions, so can extract coeff. mult. of $dx_1 \wedge dx_2$ from this.

$$d\alpha_i = \frac{\partial \alpha_i}{\partial x_1} dx_1 + \frac{\partial \alpha_i}{\partial x_2} dx_2 \quad , \quad \text{so} \quad d(\alpha) = \left(\frac{\partial \alpha_2}{\partial x_1} - \frac{\partial \alpha_1}{\partial x_2} \right) dx_1 \wedge dx_2$$

Taking this to be definition (check it is independent of local coords), then have our map. //

this is precisely the quantity in Green's thm.

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Thus our earlier exactness result says: we can find smooth function f s.t. $df = \alpha$, α : 1-form, iff $d(\alpha) = 0$.

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Finally integration of 2-forms. To make sense of integral, either restrict domain or restrict class of functions. Often want to integrate over all of surface S, so restrict functions. - use "partition of unity":

Lemma: Given $K \subseteq S$, compact subset with finite open cover $\mathcal{U}_1, \dots, \mathcal{U}_n$. $\exists$ smooth non-neg. functions $f_1, \dots, f_n$ on S with $\mathrm{supp}(f_i) \subset \mathcal{U}_i$ and $f_1 + \dots + f_n = 1$ on K (ptwise)

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Now given 2-form ρ with compact support. Say $\mathrm{supp}(\rho) = K$. Then find open cover $\{\mathcal{U}_i\}$ with $\mathcal{U}_i$'s contained in coord. charts for S. $\{f_i\}$: partition of unity. Then define

$$\int_S \rho = \sum_i \int_S f_i \rho$$

(independent of choice of functions f_i by linearity of Lebesgue integral)