On Monday, introduced de Rham cohomology: for 2-manifold (smooth)

$$\Omega_S^0 \xrightarrow{d^0} \Omega_S^1 \xrightarrow{d^1} \Omega_S^2$$

$H^0(S) = \ker(d^0)$

$H^1(S) = \ker(d^1) / \mathrm{Im}(d^0)$

$H^2(S) = \Omega_S^2 / \mathrm{Im}(d^1)$

Goal: compute these groups for all connected compact orientable smooth 2-manifolds.

For $S = S^2$, $\underbrace{H^0(S^2)}_{\text{constant functions}} \cong \mathbb{R}$, $H^1(S^2) = H^2(S^2) = 0$.

(In fact, immediate that $H^0(S) \cong \mathbb{R}$ for S connected.)

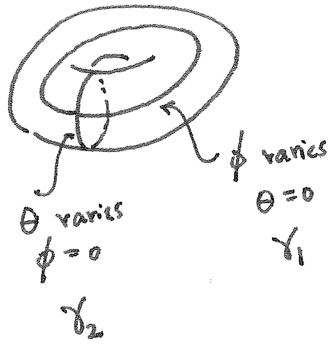
Compute $H^1(S)$. for arbitrary closed oriented surface of genus g.

(all smooth manifolds diffeomorphic to g-holed torus.)

$S = \underline{T}$: torus :

Coordinatize T with pair of angle measures $\theta, \phi \in [0, 2\pi)$

In picture:



The closed loops γ_1, γ_2 are generators for $\pi_1(T)$.

claim: $\alpha \xmapsto{\eta} \left(\int_{\gamma_1} \alpha, \int_{\gamma_2} \alpha \right)$

$$\Omega_S^1 \longrightarrow \mathbb{R}^2$$

induces an isomorphism $H^1(T) \cong \mathbb{R}^2$

① η restricts to well-defined map on $H^1(T)$. (i.e. if $\alpha \in \mathrm{Im}(d^0)$ so $\alpha = df$ for some function f, then $\eta(\alpha) = (0,0)$)

This follows from Stokes' Theorem:

$$\int_{\gamma_i} df = \int_{T \setminus \gamma_i} \underbrace{d(df)}_{0} = 0 \ ✓.$$

② $\eta : H^1(T) \to \mathbb{R}^2$ is onto.

There exist 1-forms we'll call $d\theta, d\phi$ on T. – pick $t \in T$, then give local coords θ, ϕ near t corresponding to tangent directions parallel to γ_1, γ_2.

Then $d\theta, d\phi$ in $\mathrm{Ker}(d^1)$. Are they in $\mathrm{Im}(d^0)$?

$$\int_{\gamma_1} d\theta = 0 \qquad \int_{\gamma_2} d\theta = \mathrm{length}(\gamma_2) = 2\pi. \quad \text{Similarly for } d\phi$$

(go back to definition of path integration, along γ_1 param. by t in open set, $\theta'(t)\,dt = 0$)

So not Im image of d^0

and give surjection onto $\mathbb{R}^2$. (also span T^*S_p for each $p \in$ torus)

③ η is injective. i.e. if $\alpha = P(\theta,\phi)\,d\theta + Q(\theta,\phi)\,d\phi$ is closed 1-form with $\int_{\gamma_i} \alpha = 0$ $i = 1,2$, then α is exact ($\exists f$ smooth on T s.t. $df = \alpha$)

Let $g = \int_0^\theta P(u,\phi)\,du$

Then g defines smooth function on T with $\frac{\partial g}{\partial \theta} = P(\theta,\phi)$ provided we can show $\int_0^{2\pi} P(u,\phi)\,du = 0$. True if $\phi = 0$ since $\int_{\gamma_2} \alpha = 0$ and $\int_{\gamma_2} \alpha = \int_{\gamma_2} P(\theta,\phi)\,d\theta = 0$

Why is it then true for all ϕ?

Stokes' thm. (more generally, if γ_0, γ_1 are homotopic, and α closed, then $\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha$.)

Now consider $\tilde{\alpha} = \alpha - dg$. (still closed since $d\tilde{\alpha} = d\alpha - d(dg) = 0$)

it has form $\tilde{\alpha} = \tilde{Q}(\theta, \phi)\, d\phi$. If $d(\tilde{\alpha}) = 0$, then

$$d(\tilde{Q}\, d\phi) = \frac{\partial \tilde{Q}}{\partial \theta}\, d\phi = 0 \quad \text{so} \quad \frac{d\tilde{Q}}{d\theta} \equiv 0$$ and $\tilde{Q}$ is a function of ϕ.

Same trick as before: take $g' = \int_0^\phi \tilde{Q}\, d\phi$ and then, since $\int_{\gamma_1} \tilde{\alpha} = \int_{\gamma_1} \alpha$

by Stokes' thm, then g' defines smooth function on T

with $\alpha = d(g + g')$. //

To go to higher genus surfaces, compute de Rham cohomology of cylinder C

then make higher genus surfaces by $T_1 \# C \# T_2$ (remove disk in each torus T_i, glue in cylinder along boundary of disk)

Cohomology of cylinder easy:

$C = (-1, 1) \times S^1$ with δ: circle $\{0\} \times S^1$

then $\alpha \longmapsto \int_\delta \alpha$ induces isomorphism $H^1(C) \simeq \mathbb{R}$.

So if Σ_g: genus g surface, $\alpha \longmapsto \left(\int_{\gamma_1} \alpha, \int_{\gamma_2} \alpha, \ldots \right)$ for each pair of gens. γ_1, γ_2 in each of g tori

gives isom. $H^1(\Sigma_g) \simeq \mathbb{R}^{2g}$

In fact, these integrals over loops give linear map

$$H^1(\Sigma_g) \longrightarrow \mathrm{Hom}(\pi_1(\Sigma_g, s_0), \mathbb{R})$$

which can be shown to be an isomorphism.

What about $H^2(S)$? First define variant of cohomology —

where spaces consist of forms with compact support $=: \Omega^i_{S,c}$
$i = 0, 1, 2$.